Factoring Finite-rank Linear Functional Systems*

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Abstract

By a finite-rank (linear functional) system, we mean a system of linear differential, shift and q-shift operators or, any mixture thereof, whose solution space is finite-dimensional. This poster attempts to develop an algorithm for factoring finite-rank systems into "subsystems" whose solution spaces are of lower dimension.

This work consists of three parts:

1. using module-theoretic language to interpret the factorization problem as that of finding submodules of so-called ∂-finite modules;
2. reducing the problem of finding submodules to that of finding one-dimensional submodules of exterior products of ∂-finite modules;
3. computing hyperexponential solutions of (linear functional) matrix systems to find all the one-dimensional submodules.

The first part is a generalization of the relevant material in Chapter 2 of [6]. The key ingredient of the second part is a generalization of Lemma 10 in [8] or arguments in §4.2.1 of [6], which connects d-dimensional submodules of a ∂-finite M and one-dimensional submodules of $\wedge^d M$. The third part is essentially a direct application of algorithms described in [2, 3, 5].

It should be noted that the first algorithm for factoring finite-rank partial differential systems was developed in [4, 5]. Our work is motivated by this algorithm. Although their approach is ideal-theoretic and ours is module-theoretic, these two approaches have the same origin—the associated equations method for factoring linear ode's by Beke and Schlesinger [1, 7].

Let $\mathbb{A}$ be an orthogonal Ore ring, a notion proposed recently in [2] to abstract common properties of linear differential, shift and q-shift operators. For convenience, we concern ourselves with the factorization problem of a finite-rank ideal (S) of $\mathbb{A}$ instead of a finite-rank system S.

We can extend the interpretation of the factorization problem of linear ode's as that of finding submodules of ordinary differential modules to the case of finite-rank ideals of $\mathbb{A}$. Let I be a finite-rank ideal of $\mathbb{A}$.

*This poster reports joint work with M. Bronstein and Z. Li.

There is a correspondence between finite-rank ideals of $\mathbb{A}$ which contain I and submodules of a ∂-finite module $M = \mathbb{A}/I$, i.e., an $\mathbb{A}$-module of finite dimension over the ground field. This translates the factorization problem of I into that of finding submodules of M. The goal of finding d-dimensional submodules of M is reduced to finding those one-dimensional submodules of $\wedge^d M$ whose generator is decomposable. To achieve this, we first find all one-dimensional submodules of $\wedge^d M$ through computing all hyperexponential solutions ω (in some unspecified constants) of some matrix system associated with $\wedge^d M$. Then, for each ω, we construct a map $\phi_\omega : M \to \wedge^{d+1} M$ sending v to $v \wedge \omega$. Hence, identifying the decomposability of ω is a rank computation, i.e., identifying the unspecified constants in ω such that the matrix of ϕ_ω has rank $n - d$, which is converted into solving some nonlinear systems in these constants. If ω is decomposable, then the kernel of ϕ_ω is a d-dimensional submodule of M.

Here is an outline of our algorithm for factoring finite-rank systems: given a system S of rank n, find all its subsystems of rank d.

1. construct the n-dimensional $\mathbb{A}$-module $M = \mathbb{A}/I$ with $I = (S)$;
2. compute all hyperexponential solutions ω of the matrix system associated with $\wedge^{n-d} M$;
3. test the decomposability of ω, which gives us all $(n-d)$-dimensional submodules N of M;
4. recover a rank d subsystem of S from the bases of N and I over k.

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