Over-constrained Weierstrass iteration and the nearest consistent system (summary)

Olivier Ruatta[*] Mark Sciabica[†] Agnes Szanto[‡]

Abstract

We propose a generalization of the Weierstrass iteration for over-constrained systems of equations and we prove that the proposed method allows us to find the nearest system which has at least k common roots and which is obtained via a perturbation of prescribed structure. In the univariate case we show the connection of our method to the optimization problem formulated by Karmarkar and Lakshman for the nearest GCD. In the multivariate case we generalize the expressions of Karmarkar and Lakshman, and give a simple iterative method to compute the optimum. The arithmetic complexity of the iteration is detailed.

1 Summary

In many physical and engineering applications one needs to solve over-constrained systems of equations, i.e. systems with more equations than unknowns, such that the existence of the solutions is guaranteed by some underlying physical property. However, the input system may be given only with limited accuracy due to measurement or rounding error, and thus the actual input may be inconsistent.

The work presented in this paper is concerned with the question of finding the "nearest" system with at least k distinct common roots over $\mathbb{C}$. We introduce a generalization of the Gauss-Weierstrass method [**?**]. In the univariate case, the proposed iterative method allows computation of the nearest GCD of given degree, and is closely related to the formula of Karmarkar-Lakshman for the distance to the set of systems with at least k common roots [**?**]. We show how to extend the iterative method to over-constrained systems of analytic functions. Using this extended construction we generalize the Karmarkar-Lakshman formula to the multivariate case.

[*]UNSA-INRIA, Sophia Antipolis, France
[†]Department of Mathematics, NCSU, Raleigh, NC
[‡]Department of Mathematics, NCSU, Raleigh, NC. Research is supported by NSF grant CCR-0306406

More precisely, in the *univariate case* the problem we address in the paper is the following:

Problem 1. *Given $f, g \in \mathbb{C}[x]$ and $k \in \mathbb{N}$, find a polynomial h of degree k such that there exist polynomials $\tilde{f}, \tilde{g} \in \mathbb{C}[x]$ such that h divides both $\tilde{f}$ and $\tilde{g}$ and $f - \tilde{f}$ and $g - \tilde{g}$ have prescribed supports and minimal 2-norms.*

The method proposed here is based on a generalization of the so-called Weierstrass method (also called Durand-Kerner method [**?**, **?**] or Dochev method [**?**, **?**]) introduced in [**?**] (the method was generalized successively [**?**, **?**, **?**]). We show a link of this method to the work of Karmarkar and Lakshman [**?**].

Our main result in the *univariate case* is the following theorem:

Theorem 1.1. *Let $f, g \in \mathbb{C}[x]$, $k > 0$, and $I, J \subset \mathbb{N}$. For fixed distinct $z_1, \ldots, z_k \in \mathbb{C}$ we can compute polynomials $f_{\mathbf{z}}, g_{\mathbf{z}} \in \mathbb{C}[x]$ such that the* **generalized Weierstrass iteration** *with supports I and J, defined by*

$$z_i' := z_i - \frac{f_{\mathbf{z}}'(z_i)^* f(z_i) + g_{\mathbf{z}}'(z_i)^* g(z_i)}{|f_{\mathbf{z}}'(z_i)|^2 + |g_{\mathbf{z}}'(z_i)|^2} \quad i = 1, \ldots, k, \tag{1}$$

is the Gauss-Newton iteration for finding k roots of the nearest $\tilde{f}, \tilde{g}$ with at least k common roots obtained from f, g by the perturbation of coefficients corresponding to I and J, respectively.

The polynomials $f_{\mathbf{z}}, g_{\mathbf{z}} \in \mathbb{C}[x]$ in Theorem 1.1 are given by

$$f_{\mathbf{z}}(x) := f(x) - F_{\mathbf{z},I}(x) \text{ and } g_{\mathbf{z}}(x) := g(x) - G_{\mathbf{z},J}(x)$$

where $F_{\mathbf{z},I}(x)$ (and $G_{\mathbf{z},J}(x)$ similarly) is the solution of the following generalized Lagrange interpolation problem:

Generalized Lagrange interpolation problem. *Consider $\mathbf{z} = (z_1, \ldots, z_k) \in \mathbb{C}^k$ where $z_1, \ldots, z_k$ are distinct complex numbers and some arbitrary complex numbers $f_1, \ldots, f_k \in \mathbb{C}$. Fix $I \subset \mathbb{N}$ such that $|I| \geq k$. The generalized Lagrange interpolation problem consists of finding the minimal 2-norm polynomial $F_{\mathbf{z},I} \in \mathbb{C}[x]_I$ with support I that satisfies:*

$$F_{\mathbf{z},I}(z_i) = f_i \text{ for } i = 1, \ldots, k. \tag{2}$$

The polynomial $F_{I,\mathbf{z}}(x)$ is computed using the Moore-Penrose pseudo-inverse of the following Vandermonde matrix associated with $\mathbf{z} = (z_1, \ldots, z_k)$ and $I = \{i_1, \ldots, i_p\}$:

$$V_I(\mathbf{z}) := \begin{pmatrix} z_1^{i_1} & \cdots & z_1^{i_p} \\ \vdots & \ddots & \vdots \\ z_k^{i_1} & \cdots & z_k^{i_p} \end{pmatrix}. \tag{3}$$

In the *multivariate case* the problem we address is as follows:

Problem 2. *Given an analytic function* $\vec{f} = (f_1, \ldots, f_N) : \mathbb{C}^n \to \mathbb{C}^N$, $N > n$, *and* $k > 0$. *Find perturbations* $p_1, \ldots, p_N$ *from a given finite dimensional vector space* $\mathcal{P}$ *such that* $(f_1 - p_1, \ldots, f_N - p_N)$ *have at least* k *distinct common roots in* $\mathbb{C}^n$ *and* $\|p_1\|_2^2 + \cdots + \|p_N\|_2^2$ *is minimal.*

Using a generalization of the Lagrange interpolation we obtain the following result, generalizing the formula of Karmarkar and Lakshman for the univariate nearest GCD to the multivariate case:

Theorem 1.2. *Let* $\vec{f} = (f_1, \ldots, f_N)$ *and* k *be as above. Let* $B_1, \ldots, B_N$ *be finite sets of analytic functions. Define*

$$\Omega_{\vec{B},k}(\vec{f}) := \left\{ \tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_N) \,:\, |\mathbf{V}(\tilde{f})| \geq k \text{ and } \forall i \; f_i - \tilde{f}_i \in \operatorname{span}_{\mathbb{C}}(B_i) \right\}.$$

Then the distance of $\vec{f}$ *to the set* $\Omega_{\vec{B},k}(\vec{f})$ *is equal to*

$$\min_{\vec{\mathbf{z}} \in \mathcal{R}_{\vec{B}}} \mathbf{f}_1^* M_{B_1}^{-1} \mathbf{f}_1(\vec{\mathbf{z}}) + \cdots + \mathbf{f}_N^* M_{B_N}^{-1} \mathbf{f}_N(\vec{\mathbf{z}}), \tag{4}$$

assuming that the minimum exists. Here $\vec{\mathbf{z}} = (\mathbf{z}_1, \ldots, \mathbf{z}_k) \in (\mathbb{C}^n)^k$, $\mathbf{f}_i(\vec{\mathbf{z}}) = (f_i(\mathbf{z}_1), \ldots, f_i(\mathbf{z}_k)) \in \mathbb{C}^k$, $M_{B_i}(\vec{\mathbf{z}}) \in \mathbb{C}^{k \times k}$ *is defined by*

$$M_{B_i}(\vec{\mathbf{z}}) = \left(\sum_{b \in B_i} b(\mathbf{z}_i) b(\mathbf{z}_j)^* \right)_{i,j \in \{1,\ldots,k\}}, \tag{5}$$

and $\mathcal{R}_{\vec{B}} = \{\vec{\mathbf{z}} \in (\mathbb{C}^n)^k \,:\, \forall i \; \operatorname{rank}(M_{B_i}(\vec{\mathbf{z}})) = k\}$.

Similarly as in the univariate case, in order to get an iterative method to find the minimum in (4), we need to solve the following interpolation problem:

Generalized multivariate interpolation problem: *Given* $\vec{\mathbf{z}} = (\mathbf{z}_1, \ldots \mathbf{z}_k) \in (\mathbb{C}^n)^k$, $f \in \mathbb{C}_n^\infty$ *and* $B \subset \mathbb{C}_n^\infty$ *finite set. Find* $p \in \operatorname{Span}_{\mathbb{C}}(B)$ *such that* $p(\mathbf{z}_i) = f(\mathbf{z}_i)$ *for all* $i = 1, \ldots, k$ *and* $\|p\|_B$ *is minimal. Here* $\|p\|_B$ *denotes the 2-norm of the coefficients of* p *in the* $\mathbb{C}$*-basis* B.

The polynomial $p \in \operatorname{Span}_{\mathbb{C}}(B)$ is computed using the Moore-Penrose pseudo-inverse of the generalized Vandermonde matrix associated with $B = \{b_1, \ldots, b_m\} \subset \mathbb{C}_n^\infty$ given by its entries as follows:

$$V_B(\vec{\mathbf{z}})_{s,t} := b_t(\mathbf{z}_s) \;\; s = 1, \ldots, k, \;\; t = 1, \ldots, m.$$

Finally, the main result of the paper asserts that a generalization of the Weierstrass iteration gives an iterative method to solve Problem 2:

Theorem 1.3. *Let $\vec{f} = (f_1, \ldots, f_N)$, k, and $\vec{B} = (B_1, \ldots, B_N)$ be as above. Given a fixed $\vec{\mathbf{z}} = (\mathbf{z}_1, \ldots \mathbf{z}_k) \in \mathcal{R}_{\vec{B}}$ we can find $\vec{f_{\mathbf{z}}} = (f_{\vec{\mathbf{z}},1} \ldots, f_{\vec{\mathbf{z}},N}) : \mathbb{C}^n \to \mathbb{C}^N$ such that if $J_{\vec{\mathbf{z}}}(\mathbf{x})$ is the $N \times n$ Jacobian matrix of $\vec{f_{\mathbf{z}}}(\mathbf{x})$, then the* **generalized multivariate Weierstrass iteration**, *defined by*

$$\mathbf{z}_i' := \mathbf{z}_i - J_{\vec{\mathbf{z}}}(\mathbf{z}_i)^+ \vec{f}(\mathbf{z}_i) \;\; i = 1, \ldots, k,$$

is the Gauss-Newton iteration for finding k common roots of the nearest system $\tilde{f} \in \Omega_{\vec{B},k}(\vec{f})$.