A Sister Celine Type Algorithm for Definite Summation and Integration

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Abstract. We present a new algorithm for definite symbolic summation and integration of special functions and its implementation in Mathematica.

Sister Celine's Method

Sister Celine's method [4] computes a recurrence for a sum (or multisum) $\sum_{k=0}^{\infty} f(n,k)$ where $f(n,k)$ is a given hypergeometric term. For every proper hypergeometric summand, such a recurrence exists [7], [8], [9]. The method was generalized to the q-hypergeometric case and to integrals [9]. So far, it remained restricted to the proper (q-)hypergeometric case (blue box).

Our Generalization

We generalize Sister Celine's method to sums and integrals of certain nonhypergeometric special functions. The summands (integrands) that our algorithm accepts must be elements of a difference-differential ring of height 2 that satisfies a certain technical condition. These "level-2-functions" (green box) are related to the class of ∂-finite functions [3]. Level-2-functions are closed under addition, multiplication, derivations, shifts and certain substitutions (compositions).

Just as Sister Celine's method does, our new algorithm makes an ansatz for a k-free annihilating operator for the summand $f(n,k)$ and solves a linear equation system for finding the unknown coefficients of that operator (yellow box). Sister Celine's method works only if the shiftquotients $f(n+1,k)/f(n,k)$ and $f(n,k+1)/f(n,k)$ of the summand $f(n,k)$ are rational functions in n and k; their rationality is the key for the reduction to a problem about polynomials in $Q(n)[k]$. Since our algorithm has to deal with nonhypergeometric summands as well, a different strategy for reducing to a polynomial problem is needed. The algorithm views the summand as an element of a suitably constructed ring of polynomials equipped with shifts and derivation operators. These operators are given by their action on the indeterminates of that polynomial ring.

Many classical identities found in tables [1] can be proved by our algorithm (green box).

Existence Theorem

A new theorem proves the existence of nontrivial annihilating operators and thus implies termination of the algorithm. It is a generalization of the fundamental theorem of hypergeometric summation [9].

Related Algorithms

F. Chyzak and F. Salvy [3] handle a more general class of inputs by computing Gröbner bases in the Ore algebra of linear differential-difference operators. Our algorithm is more elementary. It requires neither Gröbner basis computations, nor closure property algorithms [3].

Efficiency

Although Sister Celine's method was improved in the hypergeometric case [7, 8], it still remains too slow for many examples from applications.

Speed depends strongly on the choice of the structureset. New methods have been devised for making a good choice (yellow box).

Implementations

Existing implementations of Sister Celine's method are `MultiSum` [8] and `qMultiSum` [5] for (q-)hypergeometric summation, and `MultInt` [6] for hypergeometric integration. The new algorithm is implemented in the Mathematica package `mdf.m`, which is available upon request from the author.

Further Work

- Investigate the relation between holonomic functions and level-2-functions.
- Find a Zeilberger type algorithm [2, 10].
- Analyze the complexity of the algorithm.

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Multivariate Difference-Differential (mdd) Rings

An mdd ring is a ring equipped with derivations and shifts, which commute with each other.
Example: $\mathbb{C}(c_1,c_2,x_1)[Y_1,Y_2,Y_3]$ with derivations D_ω and D_t and a shift S_n acting as follows:

$$\begin{array}{ccccccc}
 & c_1 & c_2 & x_1 & Y_1 & Y_2 & Y_3 \\
D_\omega & 0 & 1 & 0 & 0 & 0 & -ix_1Y_3 \\
D_t & 0 & 0 & 1 & Y_2 & \frac{1}{x_1^2}\left(c_1^2-x_1^2\right)Y_1 - x_1Y_2 & -ic_2Y_3 \\
S_n & 1+c_1 & c_2 & x_1 & \frac{1}{x_1}\left(c_1Y_1-x_1Y_2\right) & \frac{1}{x_1^2}\left(\left(-c_1-c_1^2+x_1^2\right)Y_1+\left(x_1+c_1x_1\right)Y_2\right) & Y_3
\end{array}$$

Its elements correspond to analytic functions:

$$c_1 \leftrightarrow n,\quad c_2 \leftrightarrow \omega,\quad x_1 \leftrightarrow t,\quad Y_1 \leftrightarrow J_n(t),\quad Y_2 \leftrightarrow J_n'(t),\quad Y_3 \leftrightarrow \exp(-it\omega).$$

Levels

Level-1-variables: c_1, c_2, x_1, Level-2-variables: Y_1, Y_2, Y_3.

Additional Examples (new algorithm)

$$\sum_{k=0}^{n}\binom{n}{k}^2 F_k,\quad \sum_{k=0}^{n}\binom{n}{k}^2 {H_k}^2,$$

$$\sum_{n=0}^{\infty} P_n(x)z^n = \frac{1}{1-2xz+z^2},$$

$$\int_{-\infty}^{\infty} e^{-i\omega t}J_n(t)dt = 2\frac{(-i)^n T_n(\omega)}{\sqrt{1-\omega^2}}\quad (\omega^2<1)$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}P_k(z)x^m y^k,$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_0^{\infty} t^{m-n+1}J_m(at)J_n(at)dt.$$

$$\sum_{k=0}^{2n}(-1)^k q^{(n-k)^2}\begin{bmatrix}2n\\k\end{bmatrix}_q^2 S_k(q,q).$$

Examples (classical algorithm)

(q-)hypergeometric sums:

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$$

$$\sum_{i,j,k}(-1)^{i+j+k}q^{\binom{i+j+k}{2}}\begin{bmatrix}2m\\m+i\end{bmatrix}_q\begin{bmatrix}2n\\n+j\end{bmatrix}_q\begin{bmatrix}2p\\p+k\end{bmatrix}_q,$$

Generalized Sister Celine's Algorithm (Example)

Goal: compute a recurrence for

$$\int_{-\infty}^{\infty} e^{-i\omega t}J_n(t)dt.$$

The integrand $e^{-i\omega t}J_n(t)$ corresponds to Y_1Y_3 (yellow box above). Ansatz for $S=\{1,S_n,D_tS_n,S_n^2\}$:

$$A\cdot Y_1Y_3 = 0 \quad\text{where}\quad A = a_1+a_2S_n+a_3D_tS_n+a_4S_n^2,$$

with undetermined coefficients $a_1,a_2,\dots,a_4$ in $\mathbb{C}(c_1,c_2)$.
Straightforward computation in $\mathbb{C}(c_1,c_2,x_1)[Y_1,Y_2,Y_3]$ gives:

$$\begin{aligned}A\cdot Y_1Y_3 = {}& a_1Y_1Y_3 + a_2\left(\frac{c_1}{x_1}Y_1Y_3 - Y_2Y_3\right) + a_3\left(\frac{1}{x_1^2}\left(x_1^2 - ic_1c_2x_1 - c_1^2 - c_1\right)Y_1Y_3\right.\\ &\left.+\frac{1}{x_1}\left(ic_2x_1+c_1+1\right)iY_2Y_3\right) + a_4\left(\frac{1}{x_1^2}\left(x_1^2+2c_1^2+2c_1\right)Y_1Y_3 - \frac{1}{x_1}\left(c_1+1\right)Y_2Y_3\right)\end{aligned}$$

Take the numerator, and convert it to the ring $\mathbb{C}(c_1,c_2)[a_1,a_2,a_3,a_4][x_1,Y_1,Y_2,Y_3]$:

$$\begin{aligned}\text{numerator}(A\cdot Y_1Y_3) = {}&((-c_1-c_1^2)a_3+(2c_1+2c_1^2)a_4)x_1^2Y_1Y_3+(c_1a_2-ic_1c_2a_3)x_1^3Y_1Y_3+\\ &+((-2-2c_1)a_4+(1+c_1)a_3)Y_2Y_3+(a_1-a_4+a_3)x_1^4Y_1Y_3+(-a_2+ic_2a_3)Y_2Y_3.\end{aligned}$$

A sufficient condition for $A\cdot Y_1Y_3=0$ is the vanishing of the coefficient of each monomial in $[x_1,Y_1,Y_2,Y_3]$ in the numerator above:

$$(-c_1-c_1^2)a_3+(2c_1+2c_1^2)a_4=0,\quad \dots,\quad -a_2+ic_2a_3=0.$$

These equations are linear in $a_1,a_2,\dots,a_4$. Solving them over the field $\mathbb{C}(c_1,c_2)$ gives

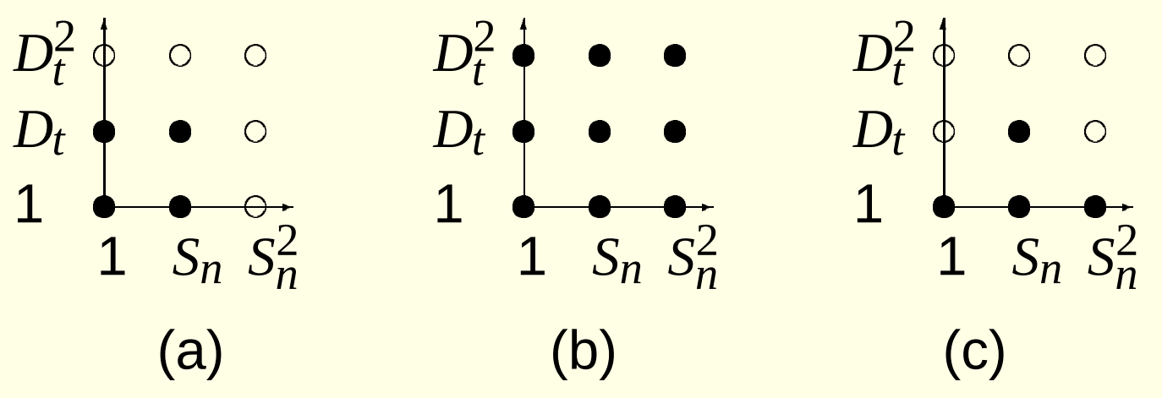
$$\left(-1+2i\omega S_n+S_n^2+2D_tS_n\right)\cdot e^{-i\omega t}J_n(t)=0.$$

Integrate:

$$\left(-1+2i\omega S_n+S_n^2\right)\cdot\int_{-\infty}^{\infty}e^{-i\omega t}J_n(t)dt=0.$$

Finding Structure Sets

Problem: How to choose S in the ansatz?



To prove that there is no annihilator on (a) for Y_1Y_3, a cheap computation in a homomorphic image $\mathbb{Z}/p\mathbb{Z}$ of $\mathbb{Q}[c_1,c_2]$ suffices, where p is a prime. A similar computation reveals that an ansatz on the structure set (b) can be reduced to an ansatz on (c).

Some level-2-functions

e^x, $\sin(x)$, $\cos(x)$, $\Gamma(x)$, Bessel functions $J_m(z)$ and $Y_m(z)$, hypergeometric functions ${}_2F_1\left[\begin{smallmatrix}a,b\\c\end{smallmatrix};z\right]$; orthogonal polynomials such as Jacobi $P_m^{(a,b)}(x)$, Gegenbauer $C_m^{(a)}(x)$, Chebyshev $T_m(x)$ and $U_m(x)$, Legendre $P_m(x)$; combinatorial sequences such as $n!$, $\binom{n}{k}$, Fibonacci numbers F_m, Harmonic numbers H_m and $H_m^{(r)}$; functions from q-analysis such as $\begin{bmatrix}m\\k\end{bmatrix}_q$, $(a;q)_m$, $[a]_q$, and generalized Fibonacci polynomials $S_m(t,q)$. New functions can be declared by their recurrences and differential equations.

The Mathematica Package mdf.m

```
In[1]:=  << "mdf.m";
```

To compute a recurrence in n for $\int_{-\infty}^{\infty} e^{-i\omega t}J_n(t)dt$, type:

```
In[2]:=  ann[
             Integrate[exp[−i ω t] BesselJ[n, t], {t, −∞, ∞}],
             {Shift[n]}]
Out[2]=  {−1 + 2 i ω Sn + Sn^2}
```

To get an ordinary differential equation, type:

```
In[3]:=  ann[
             Integrate[exp[−i ω t] BesselJ[n, t], {t, −∞, ∞}],
             {Derive[ω]}]
Out[3]=  {−i (−1 + n) (1 + n)  + 3 i ω Dω + i (−1 + ω) (1 + ω) Dω^2}
```

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