On Optimization Problem

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Optimization Problem OPTM

Let $\mathbf{R}$ = real field, $X = (x_1, x_2, \ldots, x_n)$, D a domain in $\mathbf{R}^n$, $f,\ g_i,\ i \in I,\ h_j,\ j \in J$ *polynomials* in $\mathbf{R}[X]$. $I,\ J$ and $N = \{1, 2, \ldots, n\}$ are all index sets, find optimal (i.e. greatest or least) value of $f(X)$ under constraints $h_j(X) = 0,\ g_i(X) \neq 0$ with X in D.

Let $HS = \{h_j \mid j \in J\}, GS = \{g\}, D =$ an open set O. Let $EZero_{\mathbf{R}}\{f, g, O, HS\}$ and $EVal_{\mathbf{R}}\{f, g, O, HS\}$ be the sets of extremal zeros and extremal values of the problem.

Definition 1. *Introduce Lagrange polynomial L with Lagrange multipliers $\lambda_j,\ j \in J$ by*

$$L = f + \sum_j \lambda_j \star h_j.$$

The Lagrange Polynomial Set is given by

$$LS = \{\frac{\partial L}{\partial x_i}, h_j \mid i \in N, j \in J\}.$$

Definition 2. *Consider an m-tuple of integers $t = (i_1, \ldots, i_m)$ with $1 \leq i_1 < i_2 < \ldots < i_m \leq n$, and $m =$ the number of*

elements in J. The Jacobian determinant J_t is

$$J_t = \frac{\partial(h_1, \ldots, h_m)}{\partial(x_{i_1}, \ldots, x_{i_m})}.$$

Definition 3. *Let T be the set of all such m-tuples. The Jacobian Polynomial Set is given by*

$$JS = \{J_t\,,\ h_j \mid t \in T, j \in J\}.$$

Lemma 1 For extremal zeros of above problem we have

$$EZero_{\mathbf{R}}\{f, g, O, HS\} \subset LProj\ O^+ Zero_{\mathbf{R}}(LS/g) \cup OZero_{\mathbf{R}}(JS/g). \tag{1}$$

In (1), $O^+ = O \times \mathbf{R}^m(\bigwedge)$, $\bigwedge = (\lambda_1, \lambda_2, \ldots\ldots, \lambda_m)$, and $LProj$ means the projection of $\mathbf{R}^n(X) \times \mathbf{R}^m(\bigwedge)$ on $\mathbf{R}^n(X)$.

Lemma 2 Let $f = x_1,\ HS$ an asc-set with initial-separant-product ISP and $g = ISP$. Let z be the leading variable of the first polynomial in the asc-set HS and $Proj_1$ be the projection of $\mathbf{R}^n(X) \times \mathbf{R}^m(\bigwedge)$ on the x_1-axis. For the set of extremal values we

have

$$EVal_{\mathbf{R}}\{x_1,\ ISP,\ O,\ HS\} \subset K \subset Proj_1\, OZero_{\mathbf{R}}(HS/ISP), \tag{2}$$

in which K is the finite set of real values defined by

$$K\{x_1, ISP, O, HS\} = \begin{cases} \emptyset, & z \neq x_1 \\ Proj_1\, O^+ Zero_{\mathbf{R}}(LS/ISP). & z = x_1. \end{cases} \tag{3}$$

Let $f = x_1$ and $D = O$ open, HS is an arbitrary polynomial set. By a *Modified Zero-Decomposition Theorem*, we get from HS a

set of asc-sets HS_s with initial-separant-products ISP_s such that

$$OZero_{\mathbf{R}}(HS) = \bigcup_s OZero_{\mathbf{R}}(HS_s/ISP_s). \tag{4}$$

For each s we have by *Lemma* 2

$$EVal_{\mathbf{R}}\{x_1,\ ISP_s,\ O,\ HS_s\} \subset K_s \subset Proj_1\, O^+\, Zero_{\mathbf{R}}(LS_s/ISP_s), \tag{5}$$

in which by $K_s = K\{x_1,\ ISP_s,\ O,\ HS_s\}$.

For the extremal values we have

$$EVal_{\mathbf{R}}\{x_1, \emptyset, O, HS\} \subset \bigcup_s EVal_{\mathbf{R}}\{x_1, ISP_s, O, HS_s\}.$$

Lemma 3 For the particular case of $f = x_1, GS = \emptyset, D = O$ and HS arbitrary, there is a finite set K of real values such that

$$EVal_{\mathbf{R}}\{x_1, \emptyset, O, HS\} \subset K \subset Proj_1\, O\ Zero_{\mathbf{R}}(HS). \tag{6}$$

where

$$K = K\{x_1, \emptyset, O, HS\} = \bigcup_s K\{x_1, ISP_s, O, HS_s\}, \tag{7}$$

Finite Kernel Theorem

If $f,\ g_i,\ h_j$ are all *polynomials* with $HS = \{h_j \mid j \in J\}$, $GS = \{g_i \,|\, i \in I\}$, then there is a *finite set* $K = K\{f, GS, D, HS\}$ of *real* values such that the *global* optimal value of Problem OPTM is equal to the optimal value of the finite set K, if it is known to exist.

We call the finite real value set $K = K\{f, GS, D, HS\}$ in the theorem the **Finite Kernel Set** of the Optimization Problem OPTM.

There is a package $e\_val(HS, Ord, NZ)$(by D.K.Wang) to determine the set $K = K\{f, GS, D, HS\}$.

Ex. 1. A nonlinear programming problem

$$f\,(x_1,\ x_2) = (x_1 - 2)^2 + (x_2 - 1)^2 \tag{8}$$

$$\text{Constraints}: \begin{cases} x_1 - 2x_2 + 1 & = & 0 \\ x_1^2 + 4x_2^2 - 4 & \leq & 0 \end{cases}$$

Question: Determine the least value of f w.r.t. x_1 and x_2.

Solving of Ex. 1.

1. Let x_0 be a new variable and set $h_0 = f - x_0,\ HS = \{h_0\}$, $Ord = [x_3, x_2, x_1, x_0]$ and $NZ = \emptyset$.

2. $AS := e\_val(HS,\ Ord,\ NZ)$ gives a set of 3 asc-sets AS_s:

$$[-49 + 5x_3^2 + 46x_1 + 8x_0,\ x_1 - 2x_2 + 1,\ 5x_1^2 - 18x_1 + 17 - 4x_0],$$
$$[x_3, -95 + 92x_2 + 8x_0, -49 + 46x_1 + 8x_0, 1481 - 1152x_0 + 64x_0^2],$$
$$[177 + 25x_3^2,\ -7 + 5x_2, 5x_1 - 9, -1 + 5x_0].$$

3. The Finite Kernel Set $K = K\{x_0, GS, \mathbf{R}^{n+1}, \{h_0\}\}$ is

$$K\ =\ K_1 \cup K_2 \cup K_3 = \{9 \pm \frac{23}{8}\sqrt{7}\},$$

where $K_1 = \emptyset,\ K_2 = \{9 \pm \frac{23}{8}\sqrt{7}\},\ K_3 = \emptyset$.

4. **Answer**: We have then $Least(f) = Least(x_0) = Least(K) = 9 - \frac{23}{8}\sqrt{7}$ at point$(x_1, x_2) = (\frac{-1+\sqrt{7}}{2}, \frac{1+\sqrt{7}}{4})$.