C^1 Spline Implicitization of Planar Curves

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Abstract. This poster presents a new method for constructing a low degree C^1 implicit spline representation of a given parametric planar curve. To ensure the low degree condition, quadratic B-splines are used to approximate the given curve via orthogonal projection in Sobolev spaces. Adaptive knot removal, which is based on spline wavelets, is used to reduce the number of segments. The spline segments are implicitized. After multiplying the implicit spline segments by suitable polynomial factors the resulting bivariate functions are joined along suitable transversal lines. This yields to a globally C^1 bivariate function.

Acknowledgments

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References

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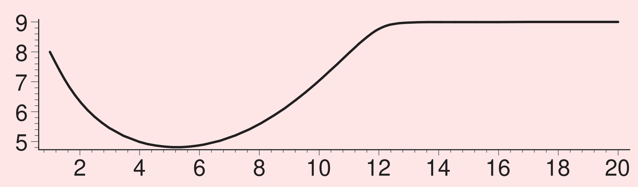
Problem

Given: A parametric curve $\mathbf{g}(t) = (x(t),\ y(t))$ of degree 15.



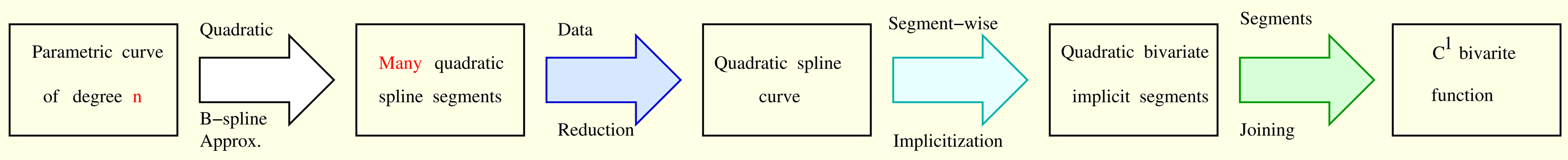
Find: The implicit form $G(x,y)$ of $\mathbf{g}(t)$.

Applications: Intersection, foot point generation, . . .

Approach A: Exact Implicitization {Resultant, Gröbner basis, . . . }

- The process is relatively complicated; resultants need evaluation of large determinants (30×30).
- Large data volume, 136 coefficients for degree 15 implicit form.
- Unwanted branches may be present.

Approach B: Low Degree Spline Implicitization {Approximate Implicitization}

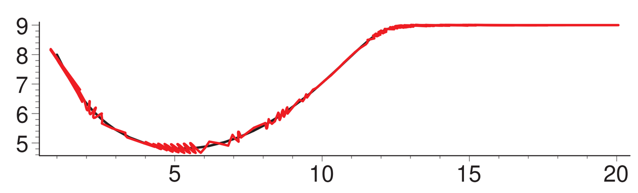


Simple computation; resultants need evaluation of determinants (4×4) only.

Small data volume, the number of the coefficients is $6 \times$ number of segments.

No unwanted branches or singularities.

Quadratic B–spline Approximation

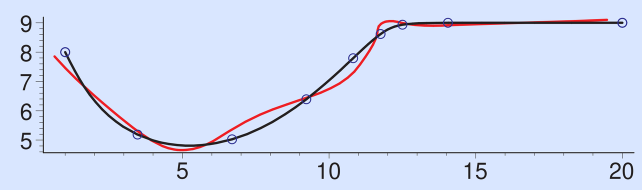


The original curve $\mathbf{g}$ (black) and the quadratic spline curve $\mathbf{g}^*$ (red). The error is exaggerated by a factor $\delta = 50000$ to make it visible.

Good approximation.

Large number of segments.

Data Reduction via wavelets



The original curve $\mathbf{g}$ (black) and the quadratic spline curve $\mathbf{g}^{**}$ (red). The error is exaggerated by a factor $\delta = 5$ to make it visible.

By setting the threshold to be 10^{-3}, the number of segments reduces from 128 to 8.

Segments–wise Implicitization

Starting from the spline curve $\mathbf{g}^{**}$, we convert the B–spline curve to Bézier curve. Then, each Bézier segment is implicitized as follows:

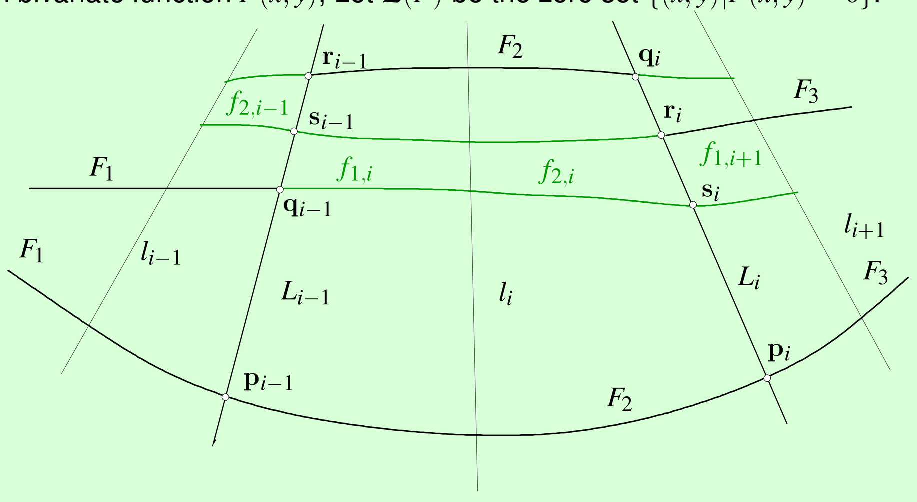
$$G(x,\ y) \ = \ \det \begin{pmatrix} Q_0(y)\,P_2(x) - P_0(x)\,Q_2(y) & Q_0(y)\,P_1(x) - P_0(x)\,Q_1(y) \\ Q_1(y)\,P_2(x) - P_1(x)\,Q_2(y) & Q_0(y)\,P_2(x) - P_0(x)\,Q_2(y) \end{pmatrix}$$

where:

$$P_i(x) = \binom{2}{i}(p_i - x), \quad Q_i(y) = \binom{2}{i}(q_i - y) \quad \text{for} \quad i = 0, 1, 2.$$

Segments Joining

In order to generate a C^1 continuous function, we modify the bivariate polynomials, which have been produced by the implicitization process, by multiplying them with suitable quadratic polynomial factors. For a given bivariate function $F(x,y)$, Let $\mathcal{Z}(F)$ be the zero set $\{(x,y)|F(x,y) = 0\}$.



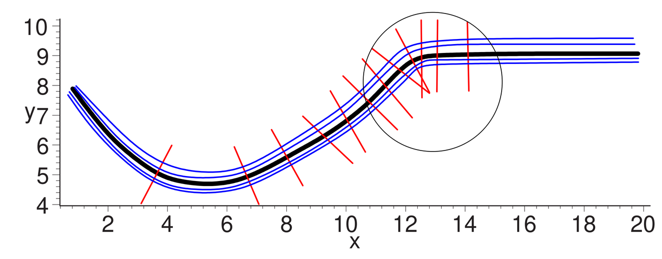
In order to obtain a *globally* C^1 spline function, we propose the following algorithm:

1. Divide the plane into patches by arbitrary lines L_i passing through the junction points $\mathbf{p}_i$. For instance, one may choose the normals to the curve.
2. Each internal patch (from the second to the $m-1$th patch, where m is the number of patches) is subdivided into two sub–patches by an arbitrary line l_i. For instance, one may choose the normal through the midpoint.
3. On the first boundary patch, we multiply by a quadratic polynomial factor $f_{2,1}$ such that $\mathcal{Z}(f_{2,1})$ is tangent continuous with the next parabola $\mathcal{Z}(G_2)$ at $\mathbf{r}_1$.
4. The next patch is modified by applying a piecewise quadratic multiplier, defined as a quadratic polynomials $f_{1,i}$ and $f_{2,i}$ on each of the two sub-patches. The multiplier has to satisfy the following conditions:
   - It must be C^1 on the whole patch.
   - $\mathcal{Z}(f_{1,i})$ is tangent continuous with the previous parabola $\mathcal{Z}(G_{i-1})$ at $\mathbf{q}_{i-1}$.
   - $\mathcal{Z}(f_{1,i})$ is tangent continuous with $\mathcal{Z}(f_{2,i-1})$ at $\mathbf{s}_{i-1}$.
   - $\mathcal{Z}(f_{2,i})$ is tangent continuous with the next parabola $\mathcal{Z}(G_{i+1})$ at $\mathbf{r}_i$.

   The first condition is fulfilled by a 7–dimensional linear space of spline functions. The other three conditions give two homogeneous linear conditions each. This results in a homogeneous system of equations, which provides at least one nontrivial solution.
5. The previous step is repeated until one arrives at the last segment.
6. Finally, a similar construction as in Step 4 is used for the last segment.

Example

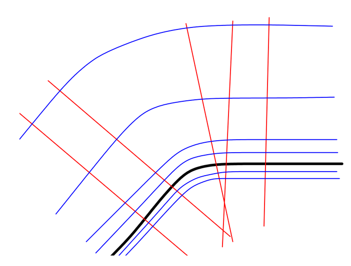
The best way to check the function C^1 continuity is the algebraic offsets.By algebraic offsets we mean: The curve $G(x,\ y) \ = \ c$ is the algebraic offset to the curve $G(x,\ y) \ = \ 0$, where c is a constant.



The piecewise quadratic implicit curve (black) and four of its algebraic offsets (blue).

Example (zoom)

In order to make the picture clearer, we enlarged a part of the curve and draw some additional algebraic offsets curves. It can clearly be seen that the algebraic offsets are C^1.



Conclusion

We have derived a method for constructing a C^1 low degree spline representation of a given parametric planar curve, which is valid within a certain neighborhood of the given curve. Compared to the existing methods for implicitization, the proposed method have the following advantages:

- The method is computationally simple. In particular, no evaluations (symbolic or numerical) of large determinants are needed.
- It produces a low degree implicit representation. For instance, the intersection of a line with the implicitized curve can be found by computing the roots of a quartic polynomial.
- The methods avoids unwanted branches or singularities, which otherwise could be present in the neighborhood of the given curve.
- The implicit function is globally C^1 continuous.
- The method is more general, it can be used to implicitize piecewise and transcendental curves.

  As a matter of future research, we plan to generalize this method to the surfaces case.