Symbolic Solution of Boundary Value Problems

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Abstract. We present a new method for solving regular BVPs for linear differential operators with constant coefficients. Unlike existing methods, it works directly on the level of operators, employing an algebraization based on a suitable noncommutative polynomial ring. The method is implemented in the *Mathematica*-based system *Theorema*.

Introduction

It is commonly believed that BVPs are a chapter of the theory of differential equations, so that one could apply symbolic methods from there. Nominally, one can indeed regard BVPs as ***inhomogeneous differential equations***.

However, such a perspective is misleading for at least two reasons:

- Conceptually: Solving differential equations means finding a ***function***, whereas solving a BVP means finding an ***operator***.
- Practically: Almost every symbolic method for differential equations needs a ***concrete term*** for the inhomogeneity.

Our novel method does justice to this fundamental difference by working ***directly on the level of operators***, thus providing an efficient as well as natural method for solving BVPs.

Problem Specification

We consider the following situation:

- Let $[a,b]$ be a finite ***interval*** in $\mathbb{R}$.
- Let T be a linear ***differential operator*** with constant coefficients having order n. We can view T as a linear operator either on the classical space $C^\infty[a,b]$ or on the distribution space $C_0^{-\infty}[a,b]$.
- Let $B_1, \ldots, B_n$ be linear ***boundary operators*** on $[a,b]$, i.e. functionals mapping a given function f to linear combinations of $f(a), \ldots, f^{n-1}(a), f(b), \ldots, f^{n-1}(b)$.

Then the ***boundary value problem*** for T and $B_1, \ldots, B_n$ is given in the upper left part of the blue frame. Our method assumes regular BVPs, meaning there is a unique solution u for all forcing functions f.

The Oblique Moore-Penrose Equalities

Let T be a linear operator between two vector spaces. Choose linear ***projectors*** P and Q onto the nullspace and range of T, respectively.

Then the oblique ***Moore-Penrose inverse*** $T^\dagger$ of T with respect to these projectors is uniquely determined by the four equations

$$\begin{aligned} T\,T^\dagger T &= T,\\ T^\dagger T\,T^\dagger &= T^\dagger,\\ T^\dagger T &= I - P,\\ T\,T^\dagger &= Q. \end{aligned}$$

Solution Algorithm

As explained above, we consider the given BVP in its more natural operator-theoretic form (see the blue frame). So we have to solve the given equation system for the so-called ***Green's operator*** G, which will yield the solution $u = G\,f$ for any forcing function f.

By some elementary ***Moore-Penrose theory*** (see the purple frame), we transform this system into a single equation as follows:

- Any oblique Moore-Penrose inverse $T^\dagger$ of T is determined by ***projectors*** to the range and nullspace of T. The former is always trivial, the latter means computing the fundamental system of the homogeneous differential equation.
- One can integrate the ***boundary conditions*** for $T^\dagger f$ by selecting an appropriate nullspace projector P out of the vector space of all such projectors.

Problem Transformation

Functional Formulation

For each $f \in C[a,b]$, find $u \in C^n[a,b]$ such that

$$\begin{aligned} T\,u &= f,\\ B_1\,u = \ldots = B_n\,u &= 0. \end{aligned}$$

$\longrightarrow$

Operator Formulation

Find $G : C[a,b] \to C^n[a,b]$ such that

$$\begin{aligned} T\,G &= 1,\\ B_1\,G = \ldots = B_n\,G &= 0. \end{aligned}$$

$\downarrow$

Moore-Penrose Equation

For $P : C[a,b] \to \mathfrak{N}(T)$ with $\mathfrak{N}(1-P)$ annihilated by $B_1, \ldots, B_n$ find G such that

$$G\,T = 1 - P.$$

$\longleftarrow$

Green's Normalization

For a right inverse $T^{\blacklozenge}$ of T and P as above, normalize the expression

$$G = (1-P)\,T^{\blacklozenge}$$

- The corresponding Moore-Penrose inverse $T^\dagger$ must coincide with the Green's operator G due to the ***regularity*** assumption.
- Usually $T^\dagger = G$ is uniquely determined by the four Moore-Penrose equations. In our situation, one can show that only the ***third equation*** given in the blue frame is necessary; the other three are redundant.

Operators Involved in BVPs

For arbitrary $u \in C^\infty[a,b]$, we have:

Differentiation

$$D\,u = u'$$

Integral and Cointegral

$$\begin{aligned} A\,u &= x \mapsto \textstyle\int_a^x u(\xi)\,d\xi\\ B\,u &= x \mapsto \textstyle\int_x^b u(\xi)\,d\xi \end{aligned}$$

Boundary Operators

$$\begin{aligned} L\,u &= x \mapsto u(a)\\ R\,u &= x \mapsto u(b) \end{aligned}$$

Multiplication Operators (for $f \in \mathfrak{F}$)

$$M_f\,u = x \mapsto f(x)\,u(x)$$

Now the problem is reduced to finding a right inverse $T^{\blacklozenge}$ of T. This can be done quite easily if we employ a suitable algebraization of the involved operators: It turns out that all of them can be reduced to linear combinations of the basic operators given in the yellow frame, leading to the concept of ***analytic polynomials***, defined in the cyan frame.

What about the Green's Function?

In principle, this already yields the desired Green's operator $G = (1-P)T^{\blacklozenge}$. However, the ***representation by polynomials*** in $\mathfrak{An}(\mathfrak{F})$ is far from unique. For example, the above right inverse would usually be written as

$$\frac{3}{4}\lceil e^{x/3}\rceil A \lceil e^{-x/3}\rceil - \frac{3}{4}\lceil e^{-x}\rceil A \lceil e^{x}\rceil,$$

using integration by parts for reducing iterated integrations.

Applying such identities in a systematic way calls for a convergent rewrite system—hence a noncommutative Gröbner basis with noetherian reduction. We have set up such a system consisting of 36 oriented polynomial identities, which we have called the ***Green's equalities***; the green frame shows some prominent examples. We have proved confluence, termination and a structural characterization of the normal form that guarantees the extraction of the Green's function g defined by

$$G\,f(x) = \int_a^b g(x,\xi)\,f(\xi)\,d\xi.$$

The following ***example*** illustrates this point. In the red frame, we have used our implementation to solve a simple BVP by the method outlined in this poster. The Green's operator for given differential operator T and the boundary operators $B_1, \ldots, B_n$ is denoted by

$$\mathrm{Green}[T, \langle B_1, \ldots, B_n\rangle],$$

and the function $\mathrm{Compute}$ issues the call for computing it. The interval $[a,b]$ is specified via the option $\mathrm{BoundaryPoints} \to \{a,b\}$.

In the literature, one can find methods for computing the Green's function via functional determinants consisting of the involved fundamental system and boundary evaluations. However, we believe that our new method has several ***advantages*** over this approach:

- As explained in the beginning, it is ***conceptually more appropriate*** to work on the operator level.

Analytic Polynomials

We use the noncommutative polynomial ring

$$\mathfrak{An}(\mathfrak{F}) = \mathbb{C}\langle A, B, D, L, R, M_f \mid f \in \mathfrak{F}\rangle,$$

which we have called the ***analytic polynomials***.

For the BVPs considered here, the domain $\mathfrak{F}$ of ***multiplier functions*** may be chosen as

$$\{x \mapsto x^k e^{\lambda x} \mid k \in \mathbb{N} \wedge \lambda \in \mathbb{C}\}.$$

We suppress the ***lambda quantifier*** for x and we write $\lceil f \rceil$ for M_f. For example, we write

$$\lceil e^{-x}\rceil A \lceil e^{4x/3}\rceil A \lceil e^{-x/3}\rceil$$

for $(3D^2 + 2D - 1)^{\blacklozenge}$.

Some Green's Equalities

Let f range over $\mathfrak{F}$.

Fundamental Theorem of Calculus

$$\begin{aligned} D\,A &= 1\\ D\,B &= -1\\ A\,D &= 1 - L\\ B\,D &= R - 1 \end{aligned}$$

Product Rule

$$\begin{aligned} D\lceil f\rceil &= \lceil f\rceil D + \lceil f'\rceil\\ A\lceil f\rceil D &= \lceil f\rceil - f(a)\,L - A\lceil f'\rceil\\ B\lceil f\rceil D &= f(b)\,R - \lceil f\rceil - B\lceil f'\rceil \end{aligned}$$

Absorption of Boundary Operators

$$\begin{aligned} L\lceil f\rceil &= f(a)\,L\\ R\lceil f\rceil &= f(b)\,L\\ L\,A = R\,B &= 0\\ R\,A = L\,B &= A + B \end{aligned}$$

- The representation of Green's operators in terms of Green's functions may ***not generalize*** to other (in particular, nonlinear) BVPs; see below.
- Regarding ***efficiency***, our algorithm should be superior as it needs determinants only for numerical rather than functional entries.

Future Work

One can naturally ask for various generalizations of the simple BVPs treated in this poster.

- We can view ***systems of differential equations***, leading to a Green's matrix instead of a Green's function and thus suggesting Gröbner bases in modules.
- The case of ***partial differential equations*** is significantly harder. One may still use a similar polynomial setup, but it is not clear how to generalize right inversion etc.
- One of the most difficult steps is towards ***nonlinear*** BVPs. A systematic approach might lead to a combination of Gröbner bases and Knuth-Bendix completion.
- We have only considered regular BVPs. If the BVP is ***underdetermined*** one can compute the so-called modified Green's operator as a suitable Moore-Penrose inverse.

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A Sample Computation

For computing the ***Green's operator*** of $(D^2 + 2D + 1)u = f$ with boundary conditions $L\,u = R\,u = 0$, we issue the following call:

```
In[13]:=  Compute[Green[D^2 + 2D + 1, ⟨L, R⟩], by → GreenEvaluator,
              EvaluatorOptions → {BoundaryPoints → {0, π}}]
```

$$\texttt{Out[13]=}\quad (1-\pi^{-1})\lceil e^{-x}x\rceil A\lceil e^{x}\rceil - \lceil e^{-x}\rceil A\lceil e^{x}x\rceil + \pi^{-1}\lceil e^{-x}x\rceil A\lceil e^{x}x\rceil - \pi^{-1}\lceil e^{-x}x\rceil B\lceil e^{x}\rceil + \pi^{-1}\lceil e^{-x}x\rceil B\lceil e^{x}x\rceil$$

Note that everything after `In[13]:=` is verbatim input, and everything after `Out[13]=` is verbatim output.

From the above Green's operator one can automatically extract the classical ***Green's function*** inducing it, namely

$$g(x,\xi) = \begin{cases} \frac{1}{\pi}\,(x-\pi)\,\xi\,e^{\xi - x} & \text{if} \quad 0 \le \xi \le x \le \pi,\\ \frac{1}{\pi}\,(\xi-\pi)\,x\,e^{\xi - x} & \text{if} \quad 0 \le x \le \xi \le \pi. \end{cases}$$