Computing Limits of Sequences

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Abstract. This poster presents a simple approach for computing limits of certain $\Pi\Sigma$-expressions using a discrete analogue of l'Hospital's rule. While details concerning termination and the treatment of special cases are still open, an implementation of the current state in the frame of the Maple system already behaves quite nice.

1 Motivation

Automated asymptotic methods has become subject of symbolic computation only recently. While systematic approaches, see, e.g., [4, 2], are available for the asymptotics of continuous functions, algorithmic treatments of the discrete case are still rare. Theoretical work like in [1] has seemingly not yet been turned into an algorithm.

This poster presents a simple method for computing limits of certain $\Pi\Sigma$-***sequences*** [3], i.e., of sequences in $\mathbb{Q}$ defined by rational functions involving (nested) product and summation quantifiers (↑ "Input Class").

Comparable to early algorithmic approaches for the continuous setting, our method makes use of some simple limit theorems, rather than on heavy theory. And comparable to these early algorithmic approaches for the continuous setting, this works surprisingly well (↑ Example, Further Examples) — although it can certainly not yet be called a satisfactory methodology.

2 Outline of the Method

The main step in our method is a discrete analogue to l'Hospital's rule which is applicable whenever the input is of the form p_n/q_n with $q_n \to \infty$ and q_n asymptotically monotonous (↑ "Mathematical Background"). The latter condition is ensured by an appropriate definition of admissible sequences (↑ "Input class"). For deciding $q_n \to \infty$, we apply a set of ad-hoc criterions (↑ "Algorithmic Details").

The repeated application of l'Hospital's rule will for sufficiently complicated input produce more and more complicated terms, and will never come to a conclusion. However, the term $\Delta p_n/\Delta q_n$ can often be reduced to a term of much simpler form. If $\Delta p_n = p_n^{(1)} + \cdots + p_n^{(r)}$ for some terms $p_n^{(i)}$, there is often one dominant term $p_n^{(i_0)}$ which "grows faster" than all the others.

We exploit this observation in a fashion similar to the reasoning in examples like

$$\lim_{n\to\infty} \frac{5n^3 + 4n^2 - 7n + 5}{3n^3 - n + 2} = \frac{5}{3}.$$

Rather than to $\Delta p_n/\Delta q_n$, the method is applied recursively to the term $\operatorname{domT}(\Delta p_n)/\operatorname{domT}(\Delta q_n)$ where $\operatorname{domT}(t)$ denotes a ***dominant term*** of t (↑ "Mathematical Background").

If the input a_n is not a sum $a_n = a_n^{(1)} + \cdots + a_n^{(r)}$ of individual terms with a unique dominant term, and it is not a product $a_n = a_n^{(1)} \cdot \cdots \cdot a_n^{(r)}$, and it can not be shown that $a_n \to \infty$ or $a_n \to -\infty$, then the method returns a failure.

3 Open Questions

An implementation of the current state already behaves quite well (↑ Example, Further Examples), but there are also open questions which remain open. Some of them are listed below.

- How can the class of admissible functions be extended to expressions involving oscillating sequences?
- How can $a_n \to \pm\infty$ be detected in a more systematic way?
- What can be said about termination?
- What can be done in cases which currently cause failures?

References

[1] Michael Boshernitzan. Discrete Orders of Infinity. In *American Journal of Mathematics*, 106:1147–1198, 1984.

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Mathematical Background

Theorem (Discrete l'Hospital's Rule) Let $(a_n)_{n=1}^\infty$, $(b_n)_{n=1}^\infty$ be sequences, and suppose that $b_n \to \infty$ $(n \to \infty)$ and that (b_n) is asymptotically strictly monotonous. Then,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\Delta a_n}{\Delta b_n},$$

where $\Delta a_n = a_{n+1} - a_n$, if the limit on the right hand side exists.

Definition (Dominant Terms[a]) If $(a_n^{(1)})_{n=1}^\infty, \dots, (a_n^{(s)})_{n=1}^\infty$ are sequences with

$$\lim_{n\to\infty} \left(\frac{a_n^{(1)}}{a_n^{(i)}} + \frac{a_n^{(2)}}{a_n^{(i)}} + \cdots + \frac{a_n^{(s)}}{a_n^{(i)}} \right) = \alpha \neq 0$$

for some $i \in \{1, \dots, s\}$, then we call

$$\operatorname{domT}(a_n^{(1)} + \cdots + a_n^{(s)}) := \alpha\, a_n^{(i)}$$

a ***dominant term***[b] of the sequence

$$a_n^{(1)} + a_n^{(2)} + \cdots + a_n^{(s)}.$$

Theorem Let (a_n) be a dominant term of a sequence (A_n) and (b_n) be a dominant term of a sequence (B_n). Then, $A_n \sim a_n$, $B_n \sim b_n$ and in particular

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} a_n \quad \text{and} \quad \lim_{n\to\infty} \frac{A_n}{B_n} = \lim_{n\to\infty} \frac{a_n}{b_n}$$

if these limits exist.

a Special case: If we write $a_n \prec b_n \iff \lim_{n\to\infty} b_n/a_n = 0$, the dominant term could be defined as the maximum term wrt. $\prec$.

b Dominant terms need not exist (consider $a_n := n + (-n)$ where $\alpha = 0$), nor will it be unique in general (consider $a_n + b_n$ with $a_n \neq b_n$ but $a_n \sim b_n$).

Input Class

- $\Pi\Sigma$-Expressions: The closure of the field $\mathbb{Q}(n)$ under field operations and indefinte products and summations with linear index bounds.
- Examples: $a^n = \prod_{k=1}^n a$ $(a \in \mathbb{Q})$, $n! = \prod_{k=1}^n k$, $\binom{2n}{n} = \frac{(2n)!}{n!^2}$, $\dfrac{\sum_{k=1}^{5n+7} \left(k! \sum_{i=1}^k \frac{1}{i^2}\right)}{\prod_{k=1}^n \left(k^2 + \sum_{i=1}^k \frac{1}{i}\right)}$, etc.
- Counterexamples: $\sum_{k=1}^{n^2} \frac{1}{k}$ (the upper bound n^2 is not linear), $\sum_{k=1}^n \binom{2n}{k}$ (the sum is not indefinite)
- Admissible Expressions: A $\Pi\Sigma$-Expression is called ***admissible,*** if the scopes of all occuring product quantifiers are asymptotically positive. Every admissible $\Pi\Sigma$-Expression corresponds to an asymptotically monotonous sequence.
- Examples: All of the examples above, except a^n if $a < 0$.

Example

INPUT: Compute $\lim_{n\to\infty} \dfrac{n^2 \mathrm{H}_{2n}}{\sum_{k=1}^n k \mathrm{H}_k}$ where $\mathrm{H}_n = \sum_{k=1}^n \frac{1}{k}$.

1 Input has the form p_n/q_n where $q_n \to \infty$.

2 Apply discrete l'Hospital, giving, after normalisation,

$$\frac{4n\,\mathrm{H}_{2n} + 7/2\,n + 4n^2\,\mathrm{H}_{2n} + 2n^2 + \mathrm{H}_{2n} + 3/2}{2n^2\,\mathrm{H}_n + 3n\,\mathrm{H}_n + 2n + \mathrm{H}_n + 1}$$

3 Determine the dominant term of the numerator:

$$\begin{aligned}
\lim_{n\to\infty} \frac{4n\,\mathrm{H}_{2n}}{7/2\,n} &= \infty &\implies&\quad 4n\,\mathrm{H}_{2n} \succ 7/2\,n\\
\lim_{n\to\infty} \frac{4n^2\,\mathrm{H}_{2n}}{4n\,\mathrm{H}_{2n}} &= \infty &\implies&\quad 4n^2\,\mathrm{H}_{2n} \succ 4n\,\mathrm{H}_{2n}\\
\lim_{n\to\infty} \frac{2n^2}{4n^2\,\mathrm{H}_{2n}} &= 0 &\implies&\quad 2n^2 \prec 4n^2\,\mathrm{H}_{2n}\\
\lim_{n\to\infty} \frac{\mathrm{H}_{2n}}{4n^2\,\mathrm{H}_{2n}} &= 0 &\implies&\quad \mathrm{H}_{2n} \prec 4n^2\,\mathrm{H}_{2n}\\
\lim_{n\to\infty} \frac{3/2}{4n^2\,\mathrm{H}_{2n}} &= 0 &\implies&\quad 3/2 \prec 4n^2\,\mathrm{H}_{2n}
\end{aligned}$$

Result: $4n^2\,\mathrm{H}_{2n}$.

4 Determine the dominant term of the denominator in the same way. Result: $2n^2\,\mathrm{H}_n$.

5 Call the algorithm recursively on

$$\frac{4n^2\,\mathrm{H}_{2n}}{2n^2\,\mathrm{H}_n} = 2\frac{\mathrm{H}_{2n}}{\mathrm{H}_n}.$$

The next level of recursion yields the result 2.

OUTPUT: $\lim_{n\to\infty} \dfrac{n^2 \mathrm{H}_{2n}}{\sum_{k=1}^n k \mathrm{H}_k} = 2.$

Algorithmic Details

We describe our method in form of a conditional rewriting system. Note that Δ is considered a map of terms to terms, rather than as a symbol of the signature.

Limit procedure

$\lim(a_n) \rightsquigarrow a_n$	if a_n is constant
$\lim(a_n \cdot b_n) \rightsquigarrow \lim(a_n)\lim(b_n)$	if not $a_n b_n \to$ "$0 \cdot \infty$"
$\lim(a_n/b_n) \rightsquigarrow \lim(\Delta a_n/\Delta b_n)$	if $b_n \to \infty$
$\lim(a_n/b_n) \rightsquigarrow \lim(a_n)/\lim(b_n)$	if $b_n \not\to \infty$
$\lim(a_n \prod_{k=1}^n b_k) \rightsquigarrow \begin{cases} \pm\infty & b_k \to \alpha > 1 \\ 0 & b_k \to \alpha \in [0,1) \\ \text{fail} & b_k \to \alpha < 0 \end{cases}$	if a_k, b_k are (bounded by) rational functions in n
$\lim(\sum_{k=1}^n a_k) \rightsquigarrow \pm\infty$	if $a_{n+1}/a_n \to \alpha > 1$ or $na_n \to \alpha \neq 0$

Simplification Rules

Besides some common normalisation rules, there is the rule

$$a_n^{(1)} + \cdots + a_n^{(r)} \rightsquigarrow \alpha\, a^{(j)}$$

where $\alpha\, a^{(j)} = \operatorname{domT}(a_n^{(1)} + \cdots + a_n^{(r)})$ reduces an addition to its dominant term. Sometimes, also "Complication rules" like

$$\sum_{k=1}^n a_k \rightsquigarrow a_n + \sum_{k=1}^{n-1} a_k \quad \text{or} \quad \frac{a_n b_n}{c_n} \rightsquigarrow \frac{b_n}{c_n/a_n}$$

turn out to be helpful.

Further Examples

A Maple implementation of our method succeeds in finding the following limits[a]

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n \left(k^2 \sum_{i=1}^k \frac{2^i}{i}\right)}{2^n n} = 4, \quad \lim_{n\to\infty} \frac{\sum_{k=1}^n \binom{3k}{k}\binom{5k}{k}}{\binom{3n}{n}\binom{5n}{n}} = \frac{84375}{83351}, \quad \lim_{n\to\infty} \frac{\sum_{k=1}^n \frac{2^k}{k^2}\binom{2k}{k}}{\left(\sum_{k=1}^n \frac{2^k}{k^2}\right)\sum_{k=1}^n \binom{2k}{k}} = \frac{3}{7}, \quad \lim_{n\to\infty} \frac{(\mathrm{H}_n)^3 \sum_{k=1}^n \frac{1}{\mathrm{H}_k}}{n \sum_{k=1}^n \frac{\mathrm{H}_k}{k}} = 2,$$

$$\lim_{n\to\infty} \frac{n^3 \left(\sum_{k=1}^n \frac{2^k}{k^2}\right)^2}{2^n \sum_{k=1}^n \frac{2^k}{k}} = 2, \quad \lim_{n\to\infty} \frac{\mathrm{H}_n \sum_{k=1}^n \frac{2^k}{k}}{n \sum_{k=1}^n \frac{2^k \mathrm{H}_k}{k^2}} = 1, \quad \lim_{n\to\infty} \frac{\sum_{k=1}^{2n} \frac{2^k k!}{k^2}}{\left(\sum_{k=1}^n \frac{4^k}{k^2}\right)\sum_{k=1}^{2n} k!} = \frac{3}{16}, \quad \lim_{n\to\infty} \frac{\sum_{k=1}^{2n} \frac{(\mathrm{H}_k)^2}{k}}{(\mathrm{H}_n)^3} = \frac{1}{3}.$$

a By $\mathrm{H}_n := \sum_{k=1}^n \frac{1}{k}$, we denote the nth harmonic number.