

Validated Numerical Methods for Systems and Control Engineering

by

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The Aim of This Research

- To investigate the development of numerical methods for systems and control which have a *guarantee on accuracy*.
- An end-product — an “*infallible*” algorithm: the user would specify *a priori* a tolerance as small as desired, and the computer would provide an answer which was guaranteed to be accurate to the specified tolerance.
- An established subject within Computer Science and a few application areas in science and engineering. Quite a new direction in the *control systems area*.

A characteristic feature — the application of *computer algebra tools* and the *avoidance of floating-point arithmetic*.

Problems Investigated

- Computation of certain quantities widely used in the modern analysis methods for control system performance, e.g. the H_2 and H_∞ norms of a stable transfer function and the induced L_∞ norm of a linear system.
- Some common algorithms for controller synthesis problems, e.g., controller computation for H_2 and H_∞ optimisation.

An Example: L_∞ Norm Computation

If the computation of a scalar quantity, e.g., the L_∞ norm, is sought, a “*validated numerical method*” will produce an *interval* of a *user-specified width* within which the *numerical answer is guaranteed to lie*.

The L_∞ norm of $G(s)$, a matrix of rational functions of the complex variable s , is defined as $\|G(s)\|_\infty := \sup_{\omega \in \mathbb{R}} \bar{\sigma}(G(j\omega))$ where $\bar{\sigma}(\cdot)$ is the largest singular value. A current implementation of the L_∞ norm computation in Matlab (a software package commonly used in systems and control area) uses floating-point arithmetic and is *prone to numerical error*. For instance, it fails to compute the L_∞ norm of

$$G(s) = \frac{s^2 + 10^{-7}s + 1}{s^2 + 10^{-8}s + 1}$$

whose L_∞ norm is $\|G(s)\|_\infty = 10$.

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[1 10^-7 1], [1 10^-8 1]))
LINFNORM iteration DID NOT converge
a lower bound for the norm is 10
out = 10.00000001492336
Inf 1.0000000000000000
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A validated numerical algorithm implemented in Maple makes use of the result below.

Theorem 1 ([1]) Let $G(s) \in L_\infty$ be rational and write its L_∞ norm $\|G(s)\|_\infty$ as γ_∞ . Let $\Phi_\gamma(s) = \gamma^2 I - G^T(-s)G(s)$ and denote $g_\gamma(s^2) = \det \Phi_\gamma(s)$. Moreover, write $g_\gamma(x) = \frac{n_\gamma(x)}{d_\gamma(x)}$. Let $h_\gamma(x)$ be the square-free part of $n_\gamma(x)$ considered as a polynomial in x and γ . Then, $\gamma > \gamma_\infty$ if and only if $\gamma > \bar{\sigma}(G(j\infty))$ and $h_\gamma(x)$ has no roots in $-\infty < x \leq 0$. Further, if γ_∞ is achieved in $0 < \omega < \infty$, then $h_{\gamma_\infty}(x)$ has a multiple root in $-\infty < x < 0$.

- The norm is equal to $\bar{\sigma}(G(0))$, $\bar{\sigma}(G(j\infty))$ or a real root of the discriminant of $h_\gamma(x)$.
- The L_∞ norm computation of $G(s)$ reduced to the *univariate polynomial real root computation problem* and the *Sturm test*.
- That is, intervals with arbitrary widths which contain these candidates can be found by means of standard real root localisation methods, for instance, *Descartes' rule of signs*.
- The *Sturm test* can determine which interval contains the true L_∞ norm by examining whether $\gamma > \bar{\sigma}(G(j\infty))$ and also the existence of roots of $h_\gamma(x)$ in $-\infty < x \leq 0$.

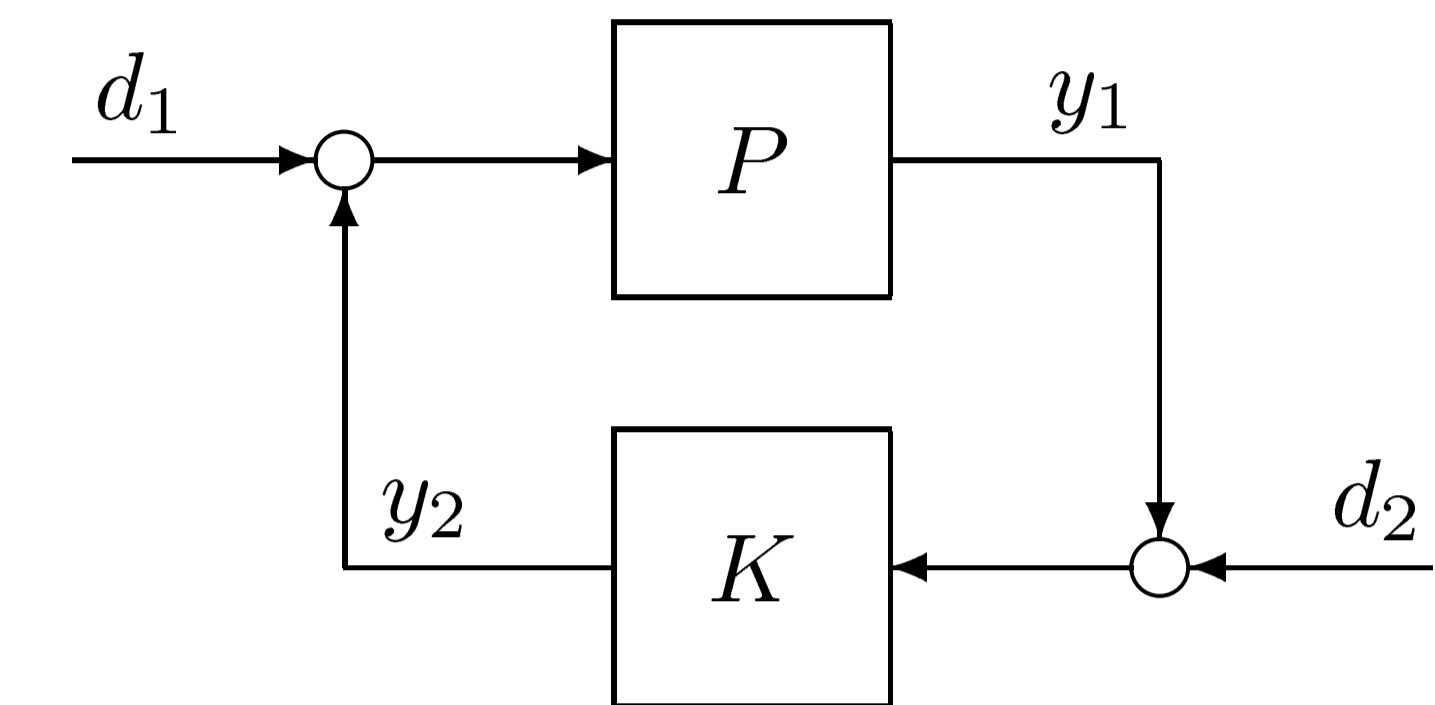
For the above example, the actual L_∞ norm is found from the discriminant of $h_\gamma(x)$, $(\gamma + 10)(\gamma - 10)(1333333333333333\gamma^2 - 1333333333333300)$.

We emphasise that the original real number data, and each step of the test, makes use of *rational number arithmetic only*. Rounding errors are avoided and the method counts as a *validated numerical method*. This contrasts with all current implementations of the L_∞ norm which can suffer from numerical error.

Controller Synthesis Problems

For instance, the H_2 optimal controller synthesis problem is formulated as: Given a plant P , find a controller K that stabilises

the closed-loop system and minimise the H_2 norm of the transfer function from $(d_1 \ d_2)^T$ to $(y_1 \ y_2)^T$.



The development of validated numerical methods for the computation of *controllers* using some standard synthesis procedures presents a number of interesting challenges.

- May not always be desirable to specify “guaranteed accuracy” in terms of the constants of the controller.
- A possibility: Find a controller whose distance from the actual controller is within a user-specified value *in terms of the standard metrics* on dynamical systems, such as the gap or ν -gap metric.
- Questions relating to *continuity of solution* to be answered along with algorithm development.
- A multiple stage algorithm may require use of *interval methods*.

Progress in this direction has been reported in [1].

Further Research

- Tackle problems which do not allow satisfactory algorithms to be implemented using ordinary floating-point arithmetic.
- Lack of reliable computational tools has been preventing some theoretical developments from being used in practice. Investigation of these problems is crucial and could have a significant impact.

References

- [1] M. Kanno. *Guaranteed Accuracy Computations in Systems and Control*. PhD thesis, University of Cambridge, 2003. In preparation.

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>> out = linfnorm(nd2sys(
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