

1 Introduction

This poster is concerned with a generalization of the notion of a variety of an ideal I , that we call a *pseudovariety* in analogy with the standard terminologies “pseudozero”[Ste96, Ste99a, Ste99b, TT94] and “pseudospectrum”[TT94, Tre99]. Other works use “pseudozero set”[HMZ03] or “root neighborhoods”[Mos86] to describe this concept.

2 Goals of this poster

- A method for visualizing low-dimensional projections of pseudovarieties
- Understanding stability of roots of nearby systems of polynomials
- Decide if nearby systems have multiple roots with respect to ϵ .

3 Pseudovarieties

For multivariate polynomials $p_i \in \mathbb{C}[x_1, \dots, x_s]$, define:

Definition 1 (Straightforward Pseudovariety).

$$PV(I_{\epsilon,p}) = \bigcup_{\|\Delta p_i\| \leq \epsilon} V(\langle p_1 + \Delta p_1, \dots, p_m + \Delta p_m \rangle)$$

where $\Delta p_i \in \mathbb{C}[x_1, \dots, x_s]$ and the norm $\|\cdot\|$ means the 2-norm of the vector of polynomial coefficients.

4 Properties of pseudovarieties

- Pseudovarieties depend on the specified generators (basis) of the ideal I .
- For $V = PV(I_{\epsilon,f_1, \dots, f_m})$ and $W = PV(I_{\epsilon,g_1, \dots, g_n})$, the following properties are valid :

$$-V \cap W = PV(I_{\epsilon,f_1, \dots, f_m, g_1, \dots, g_n}),$$

- If $\|f_i\| = 1$ and $\|g_j\| = 1$ for all i and j , then

$$V \cup W \subset PV(I_{\sqrt{2\pi}\epsilon, f, g, 1 \leq i \leq m, 1 \leq j \leq n}).$$

5 Neighborhood of polynomials

A polynomial with support $J \subset \mathbb{N}_0^s$ may be written as $p(x) = \sum_{j \in J} a_j x^j$. The tolerance associated with p is defined by a nonnegative vector $e \in \mathbb{R}^{|J|}$ whose components $e_j \geq 0$ correspond to the coefficients a_j , $j \in J$. Let $J' := \{j \in J : e_j > 0\} \neq \emptyset$ then the following defines the concept of e -neighborhood.

Definition 2 (e -neighborhood). The e -neighborhood $N(p, e)$ of the polynomial p with tolerance e consists of those polynomials $\tilde{p} \in \mathbb{C}[x_1, \dots, x_s]$, $\tilde{p}(x) = \sum \tilde{a}_j x^j$, which

satisfy

$$\|(\dots, \frac{|\tilde{a}_j - a_j|}{e_j}, \dots)\| \leq 1, \quad j \in J', \quad (1)$$

where $\tilde{a}_j = 0$, $j \notin J$ and $\tilde{a}_j = a_j$, $j \in J \setminus J'$.

6 Pseudozero criterion

Using the definition, a pseudovariety may be described as follows.

$$PV(I_{\epsilon,p}) = \bigcup_{p_i + \Delta p_i \in N(p_i, \epsilon_i)} V(\langle p_1 + \Delta p_1, \dots, p_m + \Delta p_m \rangle) \quad (2)$$

If $z \in PV(I_{\epsilon,p})$, then z is called a *pseudozero* of a system. A pseudozero criterion is described as follows.

Theorem 1 (Pseudozero criterion). $z \in \mathbb{C}^s$ is in $PV(I_{\epsilon,p})$ if and only if

$$|p_i(z)| \leq \left\| \begin{pmatrix} \vdots \\ e_j |z|^j \\ \vdots \end{pmatrix} \right\|, \quad \text{for } i = 1, \dots, m.$$

7 Method for visualization

For square systems, $m = s$, the boundary of the pseudovariety is defined by

$$F_i := p_i(z) - e^{it_i} \left\| \begin{pmatrix} \vdots \\ e_j |z|^j \\ \vdots \end{pmatrix} \right\| = 0, \quad \text{for } \begin{cases} i = 1, \dots, s, \\ 0 \leq t_i \leq 2\pi d. \end{cases}$$

where the polynomial p_i has total degree $d_i = \text{degree}(p_i)$ and the Bezout bound $d = d_1 d_2 \dots d_s$.

The method we use to visualize a pseudovariety is to find an algebraic characterization of the boundary of the pseudovariety, that we then display by a numerical parameterization. For simplicity the case $s = 2$ and $e_j = \epsilon$ for all j is explained as follows.

1. Substitute a sample point for t_2 in equations F_1 and F_2 .
2. Plug in $x_1 = u + i \cdot v$, $x_2 = w + i \cdot x$ to the equations, and obtain equations $G_1 := \text{Re}(F_1)$, $G_2 := \text{Im}(F_1)$, $G_3 := \text{Re}(F_2)$, and $G_4 := \text{Im}(F_2)$.
3. Compute the derivative of the equations G_i w.r.t. t_1 , then obtain ordinary differential equations w.r.t. t_1 .
4. Compute initial values by using Newton's method
5. Solve the initial value problem (dsolve is used here with the numeric option) (nonstiff)
6. Repeat a number of times, changing the sample point for t_2 .

Example Figures 1, 2, and 3 show projections of these parameterizations for the example

$$x^2 + y^2 - 1 = 0, \quad 25xy - 12 = 0.$$

We see from these figures that the isolated roots of the original system can merge to a double root if the perturbation is as large as $\epsilon = 0.05$.

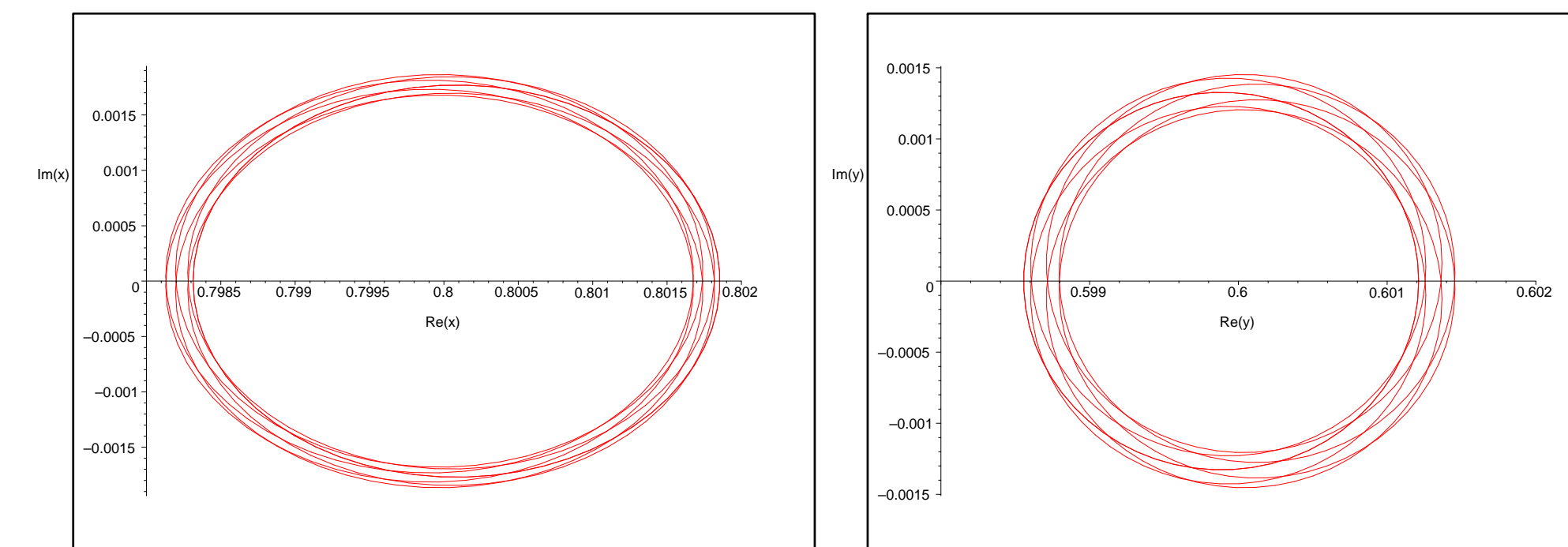


Figure 1: $\epsilon = 0.001$

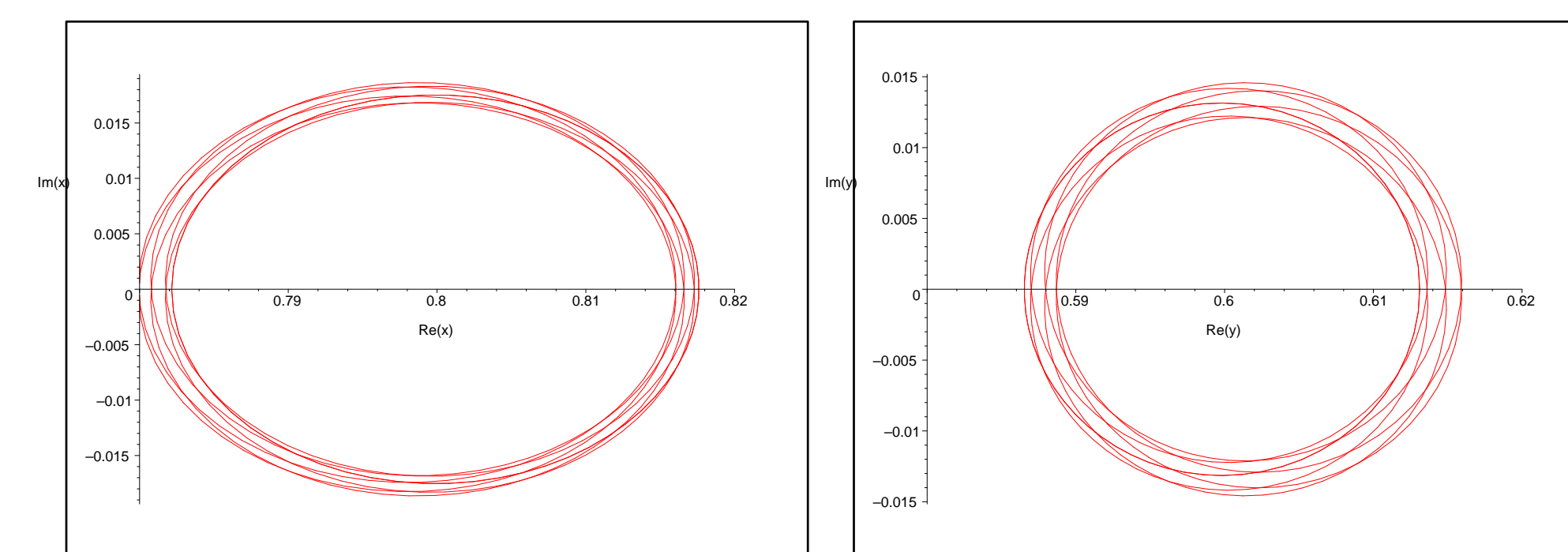


Figure 2: $\epsilon = 0.01$

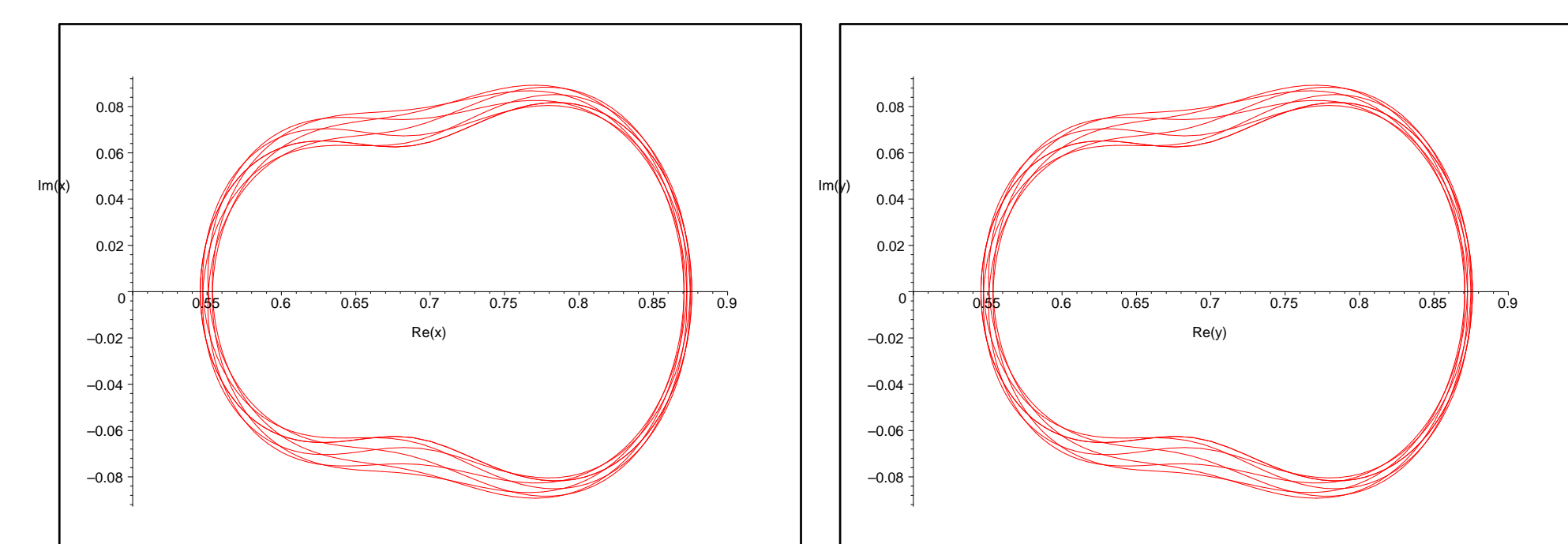


Figure 3: $\epsilon = 0.05$

8 Stability of roots

If all pseudozeros in $PV(I_{\epsilon,p})$ are isolated, e.g. Figure 1 and Figure 2, the conditioning of roots may be defined as

$$\max_{t_1, \dots, t_s} \{|x_1(t_1, \dots, t_s) - x_1^0|, \dots, |x_s(t_1, \dots, t_s) - x_s^0|\} \quad (3)$$

where (x_1^0, \dots, x_s^0) is a root of the original system p_1, \dots, p_s . (min also taken, to ensure roots correspond) This may be directly obtained from the solution of the boundaries $x_1(t_1, \dots, t_s), \dots, x_s(t_1, \dots, t_s)$.

Example The maximum perturbation of roots is bounded by

$$\begin{aligned} \max\{|x - x^0|, |y - y^0|\} &= 0.001881 \quad \text{for } \epsilon = 0.001 \\ \max\{|x - x^0|, |y - y^0|\} &= 0.020089 \quad \text{for } \epsilon = 0.01 \end{aligned}$$

9 Nearest singular systems

We observe in Figure 3 that a perturbation may change the monodromy group [CGKW02]. The critical value of ϵ for which this happens determines the nearest singular system of polynomials : typically roots will merge as

$\epsilon \rightarrow$ this critical value. We may find these critical values as follows.

$$\text{minimize } \|\Delta p_1\|^2 + \dots + \|\Delta p_s\|^2$$

subject to

$$p_i + \Delta p_i = 0, \quad i = 1, \dots, s,$$

$$\det(J(p_1 + \Delta p_1, \dots, p_s + \Delta p_s)) = 0,$$

where J is the Jacobian matrix w.r.t. polynomials $p_1 + \Delta p_1, \dots, p_s + \Delta p_s$. We may solve this problem by practical methods of optimization, e.g. Lagrange multipliers.

Example Assume the perturbation $\Delta p_1 = d_1 + d_2x + d_3y + d_4x^2 + d_5xy + d_6y^2$ and $\Delta p_2 = d_7 + d_8x + d_9y + d_{10}x^2 + d_{11}xy + d_{12}y^2$. If we could guess $(x_0, y_0) = (0.7, 0.7)$ is a nearly optimal solution, we get a system which has a multiple root at $(0.69291, 0.69291)$ by the optimization method and Newton's method.

References

- [CGKW02] Robert M. Corless, André Galligo, Ilias S. Kotsireas, and Stephen M. Watt. A geometric-numeric algorithm for absolute factorization of multivariate polynomials. In *ISSAC'2002*, pages 37–45. ACM, 2002.
- [HMZ03] J. William Hoffman, James J. Madden, and Hong Zhang. Pseudozeros of multivariate polynomials. *Math. Comp.*, 72:975–1002, 2003.
- [Mos86] Ronald G. Mosier. Root neighborhoods of a polynomial. *Math. Comp.*, 47(175):265–273, July 1986.
- [Ste96] Hans J. Stetter. Analysis of zero clusters in multivariate polynomial systems. In *ISSAC'96*, pages 127–136. ACM, 1996.
- [Ste99a] Hans J. Stetter. The nearest polynomial with a given zero, and similar problems. *SIGSAM Bull.*, 33(4):2–4, 1999.
- [Ste99b] Hans J. Stetter. Polynomials with coefficients of limited accuracy. In *Computer Algebra in Scientific Computing (Eds. V.G.Ganzha, E.W.Mayr, E.V.Vorozhtsov)*, pages 409–430. Springer, 1999.
- [Tre99] Lloyd N. Trefethen. Computation of pseudospectra. *Acta Numerica*, 8:247–295, 1999.
- [TT94] Kim-Chuan Toh and Lloyd N. Trefethen. Pseudozeros of polynomials and pseudospectra of companion matrices. *Numer. Math.*, 68:403–425, 1994.