Polynomial general solutions for first order ODEs with constant coefficients

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1 Basic notations and conceptions

Notation 1.1

K *:=differential field of meromorphic functions*

y_i *:=*$\frac{\mathrm{d}^i y}{\mathrm{d}x^i}$

K$\{y\}$ *:=differential polynomial ring*

$F(y)$ *:=irreducible differential polynomial with constant coefficients*

S *:=separant of* $F(y)$

Σ_F *:=*$\{A \in \mathbf{K}\{y\} | \, S^k A \equiv 0 \bmod \{F(y)\}\}$

$\mathcal{V}_F$ *:=differential variety of* Σ_F *over* **K**

Definition 1.2 *General solutions of* $F(y) = 0 \triangleq$ *a generic zero of* $\mathcal{V}_F$.

Definition 1.3 *Polynomial general solutions of* $F(y) = 0 \triangleq$ *general solutions of* $F(y) = 0$ *in the following form*

$$\hat{y} = \sum_{i=0}^{n} a_i x^i, (a_n \neq 0) \tag{1}$$

where a_i *are constants.*

We will abbreviate *Polynomial general solutions* to *PGSols* in the following.

2 Main results

1. A criterion theorem for whether ODEs has *PGSols* or not.

   Theorem 2.1 $F(y) = 0$ *has a PGSols iff there is a non-negative integer* k *such that the pseudo-remainder of* y_k *wrt* $F(y)$ *is zero.*

   Theorem 2.2 *Let* $F(y)$ *be of order* o. *If* $F(y) = 0$ *has a PGSols of form (1), then the* a_i *depend on* o *independent parameters.*
2. An algorithm of complexity $O(n^9)$ to get *PGSols* of first order ODEs if it exists.

3 PGSols for first order ODEs

In the following, we always assume that $F(y)$ is of order one. From the following theorem, finding *PGSols* can be reduced to finding a polynomial solution.

Theorem 3.1 *Let* $\bar{y} = \sum_{i=0}^{n} \bar{a}_i x^i$ *be a solution of* $F(y) = 0$, *where* $\bar{a}_i \in \mathbb{C}$ *and* $\bar{a}_n \neq 0$. *Then*

$$\hat{y} = \sum_{i=0}^{n} \bar{a}_i (x + c)^i \tag{2}$$

is a PGSols for $F(y) = 0$, *where* c *is arbitrary constants.*

We will solve the *PGSols* of $F(y) = 0$ by two steps:

1. Determine the degree bound of *PGSols*.
2. Find the coefficients of *PGSols*.

3.1 The degree bound of *PGSols*

Lemma 3.2 *Let* $f_1 = \sum_{i=0}^{n} \bar{a}_i x^i - y, \quad f_2 = \sum_{i=1}^{n} i \bar{a}_i x^{i-1} - y_1 \quad (n \geq 1, \, \bar{a}_n \neq 0, \, \bar{a}_i \in \mathbb{C})$. *If* $n \geq 2$, *let* R *be the Sylvester resultant of* f_1 *and* f_2 *wrt x and if* $n = 1$, *let* $R = f_2$. *Then* R *is an irreducible differential polynomial in* $\mathbb{C}[y, y_1]$ *and has the form*

$$R = (-1)^n \bar{a}_n^{n-1} y_1^n + (-1)^{n-1} n^n \bar{a}_n^n y^{n-1} + G(y, y_1) \tag{3}$$

where tdeg(G) *(total degree)* $\leq n - 1$ *and* G *does not contain the term* y^{n-1}.

Theorem 3.3 *Use the same notations in Lemma 3.2. If* $\bar{y} = \sum_{i=0}^{n} \bar{a}_i x^i$ *is a solution of* $F(y) = 0$, *then* $F(y) = \lambda R$, *where* $\lambda \in \mathbb{C}$ *and* $\lambda \neq 0$.

From Lemma 3.2 and Theorem 3.3, if $F(y) = 0$ has *PGSols* of the form (2) , $F(y)$ must be the form

$$F(y) = a y_1^n + b y^{n-1} + G(y, y_1) \tag{4}$$

where $a, b \in \mathbb{C}$ and $ab \neq 0$, and G is as in the (3).
Hence the degree of *PGSols* equals to $\deg(F(y), y_1)$.

3.2 The coefficients of *PGSols*

Lemma 3.4 *Let* $F(y)$ *be of the form (4) and have a PGSols of the form (2). Then*

$$\bar{a}_n = -\frac{b}{n^n a}. \tag{5}$$

Lemma 3.5 *Let* $F(y) = 0$ *have a PGSols of the form (2). Then we may construct a PGSols of the following form for* $F(y) = 0$

$$\hat{y} = \bar{a}_n (x + c)^n + \sum_{i=0}^{n-2} \tilde{a}_i (x + c)^i. \tag{6}$$

where $\tilde{a}_i$ *are complex numbers.*

From above Lemma, we may assume that $\bar{a}_{n-1} = 0$ in *PGSols* of the form (2).

Lemma 3.6 *Let* $F(y)$ *be of the form (4) and* $z = (-b/n^n a) x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_0$. *Substituting* y *by* z *in* $F(y)$, *the coefficients of* $x^{(n-1)^2 + i - 1}$ *in* $F(z)$ *is of the form*

$$(-b/n^n a)^{n-2} (n - 1 - i) b a_i + h_i(a_{n-1}, \cdots, a_{i+1}) \quad \text{for } i = n-2, \cdots, 0 \tag{7}$$

From Lemma 3.4,3.5 and 3.6, we may compute the coefficients of *PGSols*.

4 Algorithm and Example

Algorithm 4.1 *The input is* $F(y)$. *The output is PGSols of* $F(y) = 0$ *if it exists.*

1. If $F(y)$ can be written as the form (4), then goto step 2. Otherwise, $F(y)$ has no *PGSols* and the algorithm terminates.
2. Let $F(y)$ be of degree n in y_1. Let $\bar{a}_n = -\frac{b}{n^n a}$, $\bar{a}_{n-1} = 0$, $\bar{a}_i = -\frac{h_i(\bar{a}_{n-1}, \cdots, \bar{a}_{i+1})}{((-b/n^n a)^{n-2} (n-1-i) b)}, i = n-2, \cdots, 0$, where h_i are from Lemma 3.6. We have $\bar{a}_i \in \mathbb{C}$.
3. Let $\bar{y} = \sum_{i=0}^{n} \bar{a}_i x^i$. If $F(\bar{y}) \equiv 0$ then $\hat{y} = \sum_{i=0}^{n} \bar{a}_i (x + c)^i$ is *PGSols* of $F(y) = 0$. Otherwise, $F(y) = 0$ has no *PGSols*.

Example 4.2 *Let*

$$F = {y_1}^4 - 8\, {y_1}^3 + (6 + 24\, y)\, {y_1}^2 + 257 + 528\, y^2 - 256\, y^3 - 552\, y$$

1. *Write* F *as*

   $$F = {y_1}^4 - 256\, y^3 + G(y, y_1)$$

   where $G = -8\, {y_1}^3 + (6 + 24\, y)\, {y_1}^2 + 257 + 528\, y^2 - 552\, y$.
2. *Let* $a_4 = \frac{256}{4^4} = 1$ *and* $\bar{y} = x^4 + a_2 x^2 + a_1 x + a_0$. *Replacing* y *by* $\bar{y}$ *in* F, *the coefficients of* x^{10}, x^9, x^8 *are*

   $$384 - 256\, a_2$$
   $$-512 - 512\, a_1$$
   $$768\, a_2 + 528 - 384\, {a_2}^2 - 768\, a_0.$$

   Then we have $a_2 = \frac{3}{2}, a_1 = -1, a_0 = \frac{17}{16}$
3. *Let* $\bar{y} = x^4 + \frac{3}{2} x^2 - x + \frac{17}{16}$. *Substituting* y *by* $\bar{y}$ *into* F, F *becomes zero. Hence PGSols of* $F = 0$ *is*

   $$\hat{y} = (x + c)^4 + \frac{3}{2}(x + c)^2 - (x + c) + \frac{17}{16}$$