# Online Gröbner Basis [OGB]

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**Software name:** OGB [Online Gröbner Basis]

**Short description:** OGB calculates a (reduced, minimal) Gröbner Basis over the Rationals. Latest version (which we hope to demonstrate) allows maple syntax and calculates using any admissible term ordering following the classification of Robbiano [1]. The technologies (software) used are GMP [2], PHP* and HTML. Features include

- Sparse representation of polynomials
- Implementation of Buchberger's criteria [3] to speed up calculation
- Gröbner Bases are calculated over Q (the rationals) using arbitrary precision arithmetic (GMP [2] library)
- Software is run remotely from a web page on the server machine [4], and so uses none (or little) of the client's CPU and requires no installation!
- As far as we are aware, this is the only free software for calculating Gröbner Bases that uses none of the client's CPU!
- Since, in general, the Buchberger algorithm is not efficient, the web interface allows the user to limit time for execution

**Public access:** Remote access through web interface at **http://grobner.nuigalway.ie/**

## Abstract

### I. WHAT OGB DOES

OGB calculates in the polynomical ring $Q[x_1, x_2, ..., x_n]$, i.e the set of all polynomials in $n$ variables with rational coefficients. It does all Gröbner Basis calculations using lexicographical ordering, defined as follows: A given term $T_1 = c_1 x_1^{a_1} x_2^{a_2} ... x_n^{a_n}$ is **lexicographically greater than** another term $T_2 = c_2 x_1^{b_1} x_2^{b_2} ... x_n^{b_n}$ (and so we write $T_1 > T_2$) iff the first non-zero term in the vector $(a_1 - b_1, a_2 - b_2, ..., a_n - b_n)$ is positive. Here $c_j \in Q$, i.e. $c_j = m/n$ with $m, n \in Z$. OGB calculates the Gröbner Basis, the **minimal** Gröbner Basis (with the property that the leading monomial in polynomial $i$ is not a factor of the leading monomial in polynomial $j$ for any $i \neq j$), or the **reduced** Gröbner Basis (with the property that the leading monomial in polynomial $i$ is not a factor of any monomial in polynomial $j$ for $j \neq i$).

### II. ...YET ANOTHER GRÖBNER BASIS CALCULATOR???

This software implements known algorithms that have been implemented many other places. However the focus of OGB is on

1. Applicability to solving systems of equations: This is why the lexicographic ordering is chosen, and the reduced basis is calculated. It is known that if there are a sufficient number of equations, and if the system is solvable, the reduced basis under lexicographical ordering is always triangular and moreover the last polynomial is univariate.
2. Pedagogy: OGB is written to educate both students and academics who are not mathematicians but use mathematics (scientists, engineers,...)

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*a server-side scripting language, whose recursive acronym stands for PHP Hypertext Preprocessor [5]

3. Ease of use: OGB
   - is free !
   - requires no installation
   - is accessible on the web round the clock at `http://grobner.nuigalway.ie/`
   - is as efficient as other competitors (the only other **online** Gröbner Basis software we are aware of is at `http://www.geocities.com/CapeCanaveral/Hall/3131/main.html`. This uses `JAVA`, and runs on the clients machine; it will hence give different run-times for different users.)

## III. INTERFACE AND EXAMPLES

In designing software there is often a trade off between different types of interface: At one extreme we would "walk" the user through a series of web pages. The first might ask the number of polynomials, the second the number of terms in each polynomial, etc. This interface is clearer, but it is harder for the user to do repeated calculations. It suits the new user rather than the regular one. On the other hand we could present just one "box" into which the user enters all the polynomials, and the calculation is carried out immediately. This suits the repeated user, but not the beginner. A decision was made to construct the second type of interface. All monomials are represented sparsely: this means that for a term such as $x_1^3x_4^7 = x_1^3x_2^0x_3^0x_4^7$ we do not store variables with zero exponent (in this case we do not store $x_2$ or $x_3$)

### Example 1.

Suppose we wish to calculate with the single monomial $-13x_1^7x_3^5x_6^{32}/14$. Enter $-13, 14, 1, 7, 3, 5, 6, 32$. OGB will echo in the output (in HTML) both the polynomial(s) you have entered and the reduced Gröbner Basis. In this case it gives: `INPUT:` $-(13/14)x_1^7x_3^5x_6^{32}$ `REDUCED GRÖBNER BASIS:` $x_1^7x_3^5x_6^{32}$. Note that even when the coefficient is an integer (e.g. if it were -13 instead of -13/14) we must enter a numerator and a denominator (which will be 1 in the case of -13).

### Example 2.

Suppose we want to solve the two equations $\frac{-3}{7}x^2y^2 + xy^3 + 3 = 0$ and $xy^3 + \frac{4}{13}y^2 = 0$. Enter $-3, 7, 1, 2, 2, 2; 1, 1, 1, 1, 2, 3; 3, 1|1, 1, 1, 1, 2, 3; 4, 13, 2, 2$. Note here that a semicolon (;) separates terms and a vertical bar (|) separates polynomials. OGB gives $x_2^2 - (3501/364)$ and $x_1 + (112/3501)x_2$ from which we can easily solve the system.

## IV. LIMITATIONS

1. Only lexicographical ordering is available
2. Only works over Q (the rationals). Note however this is not a limitation in many real-world examples. We can represent an arbitrary (but finite) number of digits in a real number using rationals, e.g. write 3.572 as $\frac{3572}{1000}$.

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[1] L. Robbiano (1985), *Term orderings on the polynomial ring*, LNCS **204**, pp. 513-517.
[2] GNU Multiple Precision. This library is available free at `http://www.swox.com/gmp/`
[3] B. Buchberger (1985), *Gröbner bases: an algorithmic method in polynomial ideal theory*, in *Multidimensional Systems Theory*, edited by N. K. Bose, D. Reidel Publishing Company, 184-232.
[4] Home page for Online Gröbner Bases (OGB) is at `http://grobner.nuigalway.ie/`
[5] S. Hughes & A. Zmievski (2001), *PHP Developer's Cookbook*, Sams Publishing (2nd. edition).