QEPCAD B: A system for computing with semi-algebraic sets via cylindrical algebraic decomposition

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Software name: QEPCAD B

Short description: Performs exact, symbolic computations with semi-algebraic sets using Cylindrical Algebraic Decomposition. QEPCAD B's two most fundamental operations are:

- Simplification of formulas defining semi-algebraic sets
- Quantifier elimination for formulas defining semi-algebraic sets

Public access: http://www.cs.usna.edu/~qepcad

Abstract

QEPCAD B[1] is a system for computing with semi-algebraic sets. A *semi-algebraic set* is a subset of $\mathbb{R}^n$ that can be defined as the set of points satisfying a boolean formula combining polynomial equalities and inequalities in the variables $x_1, \ldots, x_n$. So, for example, the upper-right quadrant of the unit disk is a semi-algebraic set, since it has the defining formula

$$x_1^2 + x_2^2 \leq 1 \ \wedge \ x_1 \geq 0 \ \wedge \ x_2 \geq 0.$$

Many important problems in mathematics, science and engineering boil down to questions about semi-algebraic sets. QEPCAD B allows its users to compute with semi-algebraic sets specified by defining formulas. Computation is exact and symbolic, results being returned in the same language of defining formulas. The basic operations the system supports are *formula simplification* and *quantifier elimination*

[1] QEPCAD B extends and improves the QEPCAD system, due primarily to Hoon Hong but with contributions by many others. The author thanks Hoon Hong for permission to branch QEPCAD B off from QEPCAD, and thanks all who contributed to QEPCAD.

Quantifier Elimination: Adding quantifiers to a defining formula is, in a sense, asking a question. For example, $\exists x[x^2+bx+c=0]$ is the question "when does x^2+bx+c have a real root?" The well-known answer "when $b^2-4c \geq 0$" is an equivalent formula from which the quantified variable has been eliminated. Quantifier elimination algorithms, which produce such equivalent formulas, can be seen as providing "answers" to "questions" about semi-algebraic sets.

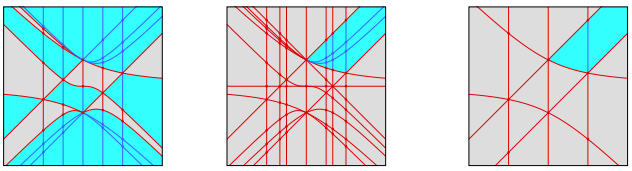
Formula Simplification: Many procedures in mathematics, performed both manually and mechanically, produce "answers" in the form of defining formulas. These defining formulas are often not particularly nice characterizations of the sets they define — hence the need for formula simplification. For example, QEPCAD B determines that the formula

$$F := 1+b^2-c^2 \geq b \wedge -c\left(b^2-c^2\right)^3+3b^2c\left(b^2-c^2\right) \vee b^2-c^2 < b$$

under the assumption $b > 0 \wedge c > 0 \wedge 1 < b+c \wedge b < 1+c \wedge c < 1+b$ is equivalent to $F' := c^2-bc-1 > 0$. Obviously, F' was a considerably better characterization for subsequent computations in the application from which this arose.

Cylindrical Algebraic Decomposition (CAD): A CAD is essentially a data-structure providing an explicit representation of a semi-algebraic set. This representation is expensive to compute, but it contains so much information about the set it represents that quantifier elimination and simplification are easily accomplished, which is why CAD is the basis for these operations in QEPCAD B. A little insight into what CAD is and how it is used is provided by the following figures, produced by QEPCAD B, which show the CAD representation for the formula F from the simplification example, followed by the CAD representing F restricted by the given assumptions, followed by the simplified CAD representation of the same set, which is what was used to provide the simplified output formula F'.

CAD for F | With assumption | Simplified CAD



This exhibit focuses on using QEPCAD B as a problem-solving tool. Examples trace problems from the application areas from which they arise, through the initial formulation of quantifier elimination or formula simplification problems, through refining problem formulations to take advantage of QEPCAD B's strengths and avoid its weaknesses. Limitations of the system and tradeoffs versus other tools are also discussed.