A Theorem for Separating Close Roots of a Polynomial and its Derivatives *

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Abstract

Let $P(z)$ be a univariate polynomial over $\mathbf{C}$, having m close roots around the origin. We present a theorem which separates a cluster of $m-k$ close roots of $\mathrm{d}^k P/\mathrm{d}z^k$ around the origin from the other roots, where $0 \le k < m$. We compare our theorem with those of Marden-Walsh and Yakoubsohn, and show superiority of our theorem.

1 Introduction

Let $P(z) = P^{(0)}(z)$ be a univariate polynomial over $\mathbf{C}$, and let $P^{(k)}(z)$ be the kth derivative of $P(z)$. There is a beautiful theorem on the roots of $P^{(k)}$, $k = 0, 1, 2, \cdots$: the convex hull containing all the roots of $P^{(k+1)}$ is covered by that containing all the roots of $P^{(k)}$. This paper presents a similar but weaker theorem for clusters of close roots.

As for the roots of $P(z)$, there are many formulas which bound the largest root and the smallest one; see [2], for example. Then, the following questions naturally arise for the close roots: under what conditions can we separate a cluster of close roots?, can we determine a disc in the complex plane, which contains only a cluster of close roots?, can we determine a similar disc for the derivatives of $P(z)$? We want to answer to these questions by using only the coefficients of the given polynomial. As for the questions on close roots of $P(z)$, Yakoubsohn [7] and Terui and Sasaki [6] obtained similar answers recently; Yakoubsohn gave a disc which separates a cluster of close roots, and Terui and Sasaki gave two discs located at the origin such that the smaller one contains a cluster of close roots and the other roots are outside the larger disc.

As for the questions on close roots of $P'(z)$, hence on those of $P^{(k)}(z)$, an old book by Marden [1] gives an answer and Yakoubsohn [7] gives another answer. In this paper, we also give an answer. As will be discussed in **3**, the old theorem imposes a very strong restriction hence it is not easy to use practically. Furthermore, the bound for close roots of $P'(z)$ is very loose. Yakoubsohn's theorem gives a much tighter bound. In this paper, we derive a new theorem using the theorem in [6].

In constructing various algorithms on univariate polynomials having close roots, we often face to decide whether or not a set of close roots are well-separated from others, what amount of scale transformation is necessary to convert a small close-roots cluster to a cluster of normal size, and so on. In such situations, the theorems as mentioned above will play important roles.

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2 Main theorem

Let $P(z)$ be the following univariate polynomial over $\mathbf{C}$.

$$P(z) = c_n z^n + \cdots + c_{m+1} z^{m+1} + z^m + e_{m-1} z^{m-1} + \cdots + e_0, \tag{2.1}$$

where the coefficients satisfy the following two conditions.

$$\begin{cases} \max\{|c_n|, \cdots, |c_{m+1}|\} = 1, \\ e \overset{\text{def}}{=} \max\{|e_{m-1}|, |e_{m-2}|^{1/2}, \cdots, |e_0|^{1/m}\} \ll 1. \end{cases} \tag{2.2}$$

$P(z)$ has m small roots around the origin (throughout this paper, we count the multiplicity). If e is small enough then the m small roots can be distinguished from others, and we cannot distinguish if e is not small. Then, what is the largest value of e that allows us to distinguish the m small roots correctly? We show a previous result which plays a crucial role in this paper (for the proof, see [6] or [5]).

Theorem 1 (Sasaki-Terui [6]) *If* $0 < e < 1/9$ *then* $P(z)$ *has* m *small roots inside a disc* D_{in} *of radius* R_{in} *and other* $n-m$ *roots outside a disc* D_{out} *of radius* R_{out}*, located at the origin, where*

$$R_{\substack{\text{out}\\ \text{in}}} = \frac{1+3e}{4} \cdot \left[1 \pm \sqrt{1 - \frac{16e}{(1+3e)^2}}\right]. \tag{2.3}$$

Corollary 1 *The following relation holds.*

$$R_{\text{in}} R_{\text{out}} = e. \tag{2.4}$$

Remark 1 *Let* $\tilde{P}(z)$ *be defined as* $\tilde{P}(z) \overset{\text{def}}{=} (z^n/e^m)\, P(e/z)$*.* $\tilde{P}(z)$ *has* $n-m$ *small roots around the origin, and the discs for* $\tilde{P}(z)$ *are the same as those for* $P(z)$*;* $n-m$ *small roots of* $\tilde{P}(z)$ *are inside of the disc* D_{in} *and other* m *roots are outside of the disc* D_{out}*.*

Figure 1 shows the e-dependence of R_{in} and R_{out}.

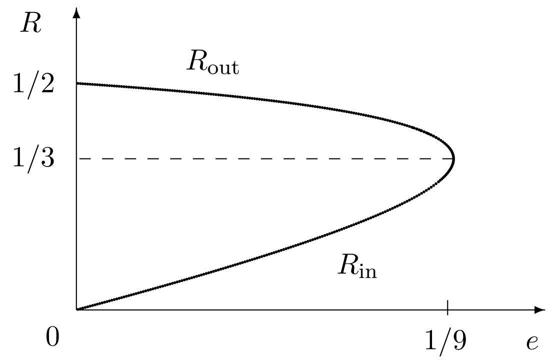

Fig.1 e-dependence of R_{in} and R_{out}.

Now, we consider the derivative of $P(z)$:

$$P'(z) = nc_n z^{n-1} + \cdots + (m+1) c_{m+1} z^m + m z^{m-1} + (m-1) e_{m-1} z^{m-2} + \cdots + e_1. \tag{2.5}$$

We transform $P'(z)$ to $\hat{P}'(z)$ as follows.

$$P'(z) \;\mapsto\; \hat{P}'(z) = (\gamma^{m-1}/m)\, P'(z/\gamma), \tag{2.6}$$

where the scale factor γ is determined as

$$\gamma \;=\; \max\Big\{ \Big(\frac{m{+}1}{m}\,|c_{m+1}|\Big)^{1/1},\; \Big(\frac{m{+}2}{m}\,|c_{m+2}|\Big)^{1/2},\; \cdots,\; \Big(\frac{n}{m}\,|c_n|\Big)^{1/(n-m)} \Big\}. \tag{2.7}$$

Then, $\hat{P}'(z)$ becomes the same form polynomial as $P(z)$:

$$\begin{aligned} \hat{P}'(z) \;&=\; \hat{c}'_{n-1}z^{n-1} + \cdots + \hat{c}'_m z^m + z^{m-1} + \hat{e}'_{m-2}z^{m-2} + \cdots + \hat{e}'_0, \\ &\quad \max\{|\hat{c}'_{n-1}|,\, |\hat{c}'_{n-2}|,\, \cdots,\, |\hat{c}'_m|\} = 1. \end{aligned} \tag{2.8}$$

We define e' similarly as e:

$$\begin{aligned} e' \;&\stackrel{\text{def}}{=}\; \max\{|\hat{e}'_{m-2}|,\, |\hat{e}'_{m-3}|^{1/2},\, \cdots,\, |\hat{e}'_0|^{1/(m-1)}\} \\ &=\; \gamma \max\Big\{ \Big(\tfrac{m-1}{m}\,|e_{m-1}|\Big)^{1/1},\; \Big(\tfrac{m-2}{m}\,|e_{m-2}|\Big)^{1/2},\; \cdots,\; \Big(\tfrac{1}{m}\,|e_1|\Big)^{1/(m-1)} \Big\}. \end{aligned} \tag{2.9}$$

Lemma 1 *The following inequalities hold.*

$$\Big(\frac{n}{m}\Big)^{1/(n-m)} \;\le\; \gamma \;\le\; \frac{m{+}1}{m}, \tag{2.10}$$

$$0 \;\le\; e' \;\le\; \Big(\frac{m{-}1}{m}\Big)\gamma e. \tag{2.11}$$

Proof. We first show the following inequalities for integer j.

$$1)\quad \Big(\frac{m{-}j}{m}\Big)^{1/j} > \Big(\frac{m{-}j{-}1}{m}\Big)^{1/(j+1)} \quad \text{for}\;\; j < m{-}1, \tag{2.12}$$

$$2)\quad \Big(\frac{m{+}j}{m}\Big)^{1/j} > \Big(\frac{m{+}j{+}1}{m}\Big)^{1/(j+1)} \quad \text{for}\;\; j \ge 1. \tag{2.13}$$

1) is equivalent to $(m{-}j)^{j+1} > m\,(m{-}j{-}1)^j$. This inequality is proved as

$$\begin{aligned} &(m{-}j{-}1{+}1)^{j+1} = (m{-}j{-}1)^{j+1} + (j{+}1)\,(m{-}j{-}1)^j + \cdots + 1 \\ &>\; (m{-}j{-}1)^{j+1} + (j{+}1)\,(m{-}j{-}1)^j = m\,(m{-}j{-}1)^j. \end{aligned}$$

2) is equivalent to $(m{+}j)^{j+1} > m\,(m{+}j{+}1)^j$. Since we have

$$(m{+}j)^{j+1} = (m{+}j{+}1)^{j+1} - (j{+}1)\,(m{+}j{+}1)^j + {}_{j+1}C_2\,(m{+}j{+}1)^{j-1} - \cdots,$$

inequality 2) is equivalent to

$${}_{j+1}C_2\,(m{+}j{+}1)^{j-1} - {}_{j+1}C_3\,(m{+}j{+}1)^{j-2} + {}_{j+1}C_4\,(m{+}j{+}1)^{j-3} - \cdots \;>\; 0.$$

For $j = 1$, this inequality is $1 > 0$. For $j \ge 2$, we pair two successive terms of the left-hand-side expression from the left (for odd j, the last term 1 remains unpaired), then we see that each pair is positive because ${}_{j+1}C_{j'+1} = {}_{j+1}C_{j'} \cdot (j{+}1{-}j')/(j'+1)$ and $(m{+}j{+}1) > (j{+}1{-}j')/(j'+1)$. Hence, we have 2).

Now, we prove the lemma. The definition of γ and the condition $\max\{|c_n|, \cdots, |c_{m+1}|\} = 1$ imply $\gamma \le \max\{(\frac{m+1}{m})^{1/1}, \cdots, (\frac{n}{m})^{1/(n-m)}\}$, and $\gamma \ge (\frac{m+j}{m})^{1/j}$ for such j that $|c_{m+j}| = 1$. Hence, 1) gives (2.10). The extreme case $e_{m-1} = \cdots = e_1 = 0$ ($|e_0| = e^m$) implies $e' = 0$, and the definition of e' implies $e' \le \gamma e \max\{(\frac{m-1}{m})^{1/1}, \cdots, (\frac{1}{m})^{1/(m-1)}\}$. Hence, 2) gives the r.h.s. of (2.11). □

Theorem 2 (main theorem) *Let $e < 1/9$ and e' be defined as above. Then, $P'(z)$ has $m-1$ small roots inside a disc $D'_{\rm in}$ of radius $R'_{\rm in}$ and other $n-m$ roots outside a disc $D'_{\rm out}$ of radius $R'_{\rm out}$, located at the origin, where*

$$R'_{\substack{\rm out\\ \rm in}} = \frac{1+3e'}{4\gamma}\cdot\Big[1\pm\sqrt{1-\frac{16e'}{(1+3e')^2}}\Big]. \tag{2.14}$$

The radii $R'_{\rm in}$ and $R'_{\rm out}$ satisfy the following equality and inequalities.

$$R'_{\rm in}R'_{\rm out} = e'/\gamma^2, \tag{2.15}$$

$$2\Big(\frac{m}{m+1}\Big)e' \;<\; R'_{\rm in} \;\le\; \Big(\frac{m}{n}\Big)^{1/(n-m)}R_{\rm in}, \tag{2.16}$$

$$\Big(\frac{m}{m+1}\Big)^2 e' \;<\; R'_{\rm in}R'_{\rm out} \;<\; \Big(\frac{m-1}{m}\Big)\Big(\frac{m}{n}\Big)^{1/(n-m)}e, \tag{2.17}$$

$$\frac{1}{3}\Big(\frac{m}{m+1}\Big) \;<\; R'_{\rm out} \;\le\; \Big(\frac{m}{n}\Big)^{2/(n-m)}\Big(\frac{R_{\rm in}}{R'_{\rm in}}\Big)R_{\rm out}. \tag{2.18}$$

Proof. The r.h.s. inequalities in (2.10) and (2.11) imply $e' \le e\,(m^2-1)/m^2 < e$, hence we can apply Theorem 1 to $\hat{P}'(z)$. The root $\hat{\zeta}$ of $\hat{P}'(z)$ corresponds to the root $\hat{\zeta}/\gamma$ of $P'(z)$, hence we obtain (2.14). Let $\hat{R}'_{\rm in}$ and $\hat{R}'_{\rm out}$ be radii of the discs for $\hat{P}'(z)$, hence $R'_{\rm in} = \hat{R}'_{\rm in}/\gamma$ and $R'_{\rm out} = \hat{R}'_{\rm out}/\gamma$. Then, we obtain (2.15) from (2.4).

Since $R_{\rm in}$ ($R_{\rm out}$, resp.) is monotone increasing (decreasing, resp.) with respect to e, so long as $e < 1/9$, we have $\hat{R}'_{\rm in} < R_{\rm in}$ or $R'_{\rm in} < R_{\rm in}/\gamma$, hence (2.10) gives the r.h.s. inequality of (2.16). Next, (2.15) and $\hat{R}'_{\rm out} \le 1/2$ imply $R'_{\rm in} \ge 2e'/\gamma$, hence (2.10) gives the l.h.s. inequality of (2.16). Similarly, we obtain (2.17) from relations $R'_{\rm in}R'_{\rm out} = e'/\gamma^2 \le (\frac{m-1}{m})\,e/\gamma$ and inequalities in (2.10). Finally, $\frac{1}{3} < \hat{R}'_{\rm out} = \gamma R'_{\rm out}$ gives the l.h.s. of (2.18), and the inequality $e' < e$ gives $\gamma^2 R'_{\rm in}R'_{\rm out} < R_{\rm in}R_{\rm out}$ hence we obtain the r.h.s. of (2.18) from (2.10). □

Corollary 2 *Let $P^{(k)}(z)$ be the k-th derivative of $P(z)$. If $e < 1/9$ then, for $k = 1, 2, \cdots, m-1$, there exist discs $D^{(k)}_{\rm in}$ and $D^{(k)}_{\rm out}$, of radii $R^{(k)}_{\rm in}$ and $R^{(k)}_{\rm out}$, located at the origin, such that $R_{\rm in} > R^{(1)}_{\rm in} \ge \cdots \ge R^{(m-1)}_{\rm in}$ (equality $R^{(j)}_{\rm in} = R^{(j+1)}_{\rm in}$ holds only when $R^{(j)}_{\rm in} = R^{(j+1)}_{\rm in} = 0$), and that all the $m-k$ small roots of $P^{(k)}(z)$ are contained in $D^{(k)}_{\rm in}$ and other $n-m$ roots are outside of $D^{(k)}_{\rm out}$. The radii $R^{(k)}_{\rm in}$ and $R^{(k)}_{\rm out}$ satisfy inequalities*

$$R^{(k)}_{\rm in} \;<\; \Big[\frac{m(m-1)\cdots(m-k+1)}{n(n-1)\cdots(n-k+1)}\Big]^{1/(n-m)}R_{\rm in}, \tag{2.19}$$

$$R^{(k)}_{\rm out}R^{(k)}_{\rm in} \;<\; \Big(\frac{m-k}{m}\Big)\Big[\frac{m(m-1)\cdots(m-k+1)}{n(n-1)\cdots(n-k+1)}\Big]^{1/(n-m)}e. \tag{2.20}$$

Remark 2 *One may expect inequality $R'_{\rm out} < R_{\rm out}$ (or $R'_{\rm out} > R_{\rm out}$), but this is wrong. We first note that there is a case that $\gamma \simeq 1$: if $c_{n-1} = \cdots = c_{m+1} = 0$ then (2.7) tells us that $\gamma = (\frac{n}{m})^{1/(n-m)} \longrightarrow 1$ as $n \to \infty$. With this in mind, consider an extreme case that $e_{m-1} = \cdots = e_1 = 0$ hence $e' = 0$, then we have $R'_{\rm out} = \hat{R}'_{\rm out}/\gamma \simeq \hat{R}'_{\rm out} = 1/2 > R_{\rm out}$. On the other hand, we often have $R'_{\rm out} < R_{\rm out}$. For example, consider the case that $|c_{m+1}| = 1$ and $e = e_{m-1}$, then we have $\gamma = \frac{m+1}{m}$ and $e' = \gamma\frac{m-1}{m}e = \frac{m^2-1}{m^2}e$. For large m, we have $\gamma = 1 + O(1/m)$ and $e'/e = 1 - O(1/m^2)$. Hence, Fig. 1 and relation $R'_{\rm out} = \hat{R}'_{\rm out}/\gamma$ tell us that we will have $R'_{\rm out} < R_{\rm out}$ unless $e \approx 1/9$.*

We check sharpness of our formulas by examples. First, we note that there is a polynomial, let it be $P_B(z) = (z-\zeta_1)\cdots(z-\zeta_n)$ with $|\zeta_1| \le \cdots \le |\zeta_m| < |\zeta_{m+1}| \le \cdots \le |\zeta_n|$, such that $|\zeta_m| \to R_{\rm in}$ and $|\zeta_{m+1}| \to R_{\rm out}$ in the limit $m \to \infty$ and $n-m \to \infty$. The polynomial is

$$P_B(z) = z^n + \cdots + z^{m+1} - z^m + ez^{m-1} + \cdots + e^m.$$

In fact, suppose P_B has a real root ζ, $e < |\zeta| < 1$, then

$$\begin{aligned} P_B(\zeta) &= \zeta^{m+1}\Big(\frac{1-\zeta^{n-m}}{1-\zeta}\Big) - \zeta^m + e\zeta^{m-1}\Big(\frac{1-(e/\zeta)^m}{1-e/\zeta}\Big) \\ &\simeq \zeta^m\Big(\frac{\zeta}{1-\zeta} + \frac{e}{\zeta-e} - 1\Big) \quad \text{if } m \gg 1 \text{ and } n-m \gg 1. \end{aligned}$$

Hence, $P_B(\zeta) = 0$ gives $\zeta/(1-\zeta) + e/(\zeta-e) \simeq 1 \Longrightarrow \zeta \simeq R_{\rm in}$ or $\zeta \simeq R_{\rm out}$.

Next, by changing k as $k = 0 \Rightarrow 1 \Rightarrow 2 \Rightarrow \cdots$, we observe shrinking of a cluster of close roots of $P^{(k)}(z)$. The sample polynomial is

$$P(z) = z^{10} - z^8/2 - z^7/3 + z^5 + ez^4 - e^2z^3 + e^4z/4 + e^5, \quad e = 0.1.$$

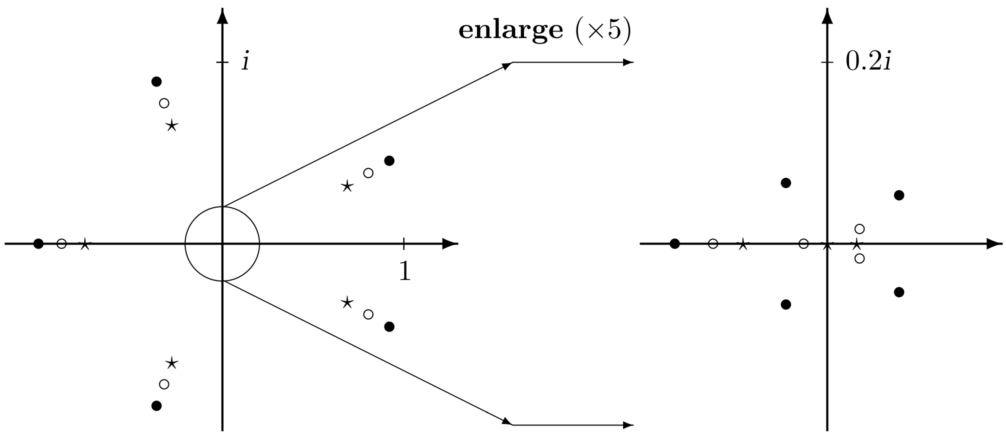

Fig. 2 Distribution of the roots of $P^{(k)}$ $(k = 0, 1, 2)$.
($\bullet$ for $P(z)$, $\circ$ for $P^{(1)}(z)$, and $\star$ for $P^{(2)}(z)$)
The left for larger roots, and the right for the close roots.

Figure 2 shows the distribution of the roots. The disc radii are: $R_{\rm out} = 0.4$, $R^{(1)}_{\rm out} \simeq 0.364$, $R^{(2)}_{\rm out} \simeq 0.322$, and $R_{\rm in} = 0.25$, $R^{(1)}_{\rm in} \simeq 0.191$, $R^{(2)}_{\rm in} \simeq 0.134$. Generally speaking, an upper bound (lower bound, resp.) is often an overestimate (underestimate, resp.). However, these radii indicate that the close roots and the other roots are separated by discs $D^{(k)}_{\rm out}$ and $D^{(k)}_{\rm in}$ fairly well.

Remark 3 *Converting $P(z)$ to an optimal form by shifting the origin, we will be able to obtain a tighter bound for the close roots. This optimization usually requires an iterative process, and most bound formulas do not include such an optimization process.*

3 Comparison with other theorems

We compare our theorem with those of Marden-Walsh and Yakoubsohn.

Theorem 3 (Marden-Walsh [1]) *Let $D(z_0, r)$ denote a disc of radius r, located at $z = z_0$ in the complex plane. Suppose that $P(z)$ has m roots in a disc $D(z_0, \tilde{R}_{\rm in})$ and other $n-m$ roots outside a disc $D(z_0, \tilde{R}_{\rm out})$, where $\tilde{R}_{\rm in} < \tilde{R}_{\rm out}$. Then, if*

$$\frac{\tilde{R}_{\rm out} + \tilde{R}_{\rm in}}{\tilde{R}_{\rm in}} > \frac{2n}{m}, \tag{3.1}$$

$P'(z)$ has $m-1$ roots in $D(z_0, \tilde{R}_{\rm in})$ and other $n-m$ roots outside of $D(z_0, \tilde{R}'_{\rm out})$, where

$$\tilde{R}'_{\rm out} = \left(\frac{m}{n}\right)(\tilde{R}_{\rm out} + \tilde{R}_{\rm in}) - \tilde{R}_{\rm in}. \tag{3.2}$$

We first note that Theorem 3 does not tell us in which case the condition (3.1) holds. If we must compute the roots of $P(z)$ to know the values of $\tilde{R}_{\rm in}$ and $\tilde{R}_{\rm out}$, then Theorem 3 will be practically useless. Therefore, Theorem 3 is less complete than Theorem 2. Comparing Theorem 3 with Theorem 2, we see that the condition (3.1) is very restrictive if $n \gg m$ (which is often the case). Furthermore, $\tilde{R}'_{\rm out}$ is much smaller than $\tilde{R}_{\rm out}$; in fact, if $\tilde{R}_{\rm out}$ is so chosen as $(\tilde{R}_{\rm out} + \tilde{R}_{\rm in})/\tilde{R}_{\rm in} \simeq 2n/m$, then $\tilde{R}'_{\rm out} \simeq \tilde{R}_{\rm in}$. Hence, $\tilde{R}'_{\rm out}$ in (3.2) is a too small estimation. On the other hand, Theorem 2 tells us that $R'_{\rm out}$ does not strongly depends on n/m.

Theorem 4 (Yakoubsohn [7], see also [8]) *Let $P(z)$ have m close roots around $z = z_0$, where $m > 1$. Let $D(z_0, r)$ be a disc of radius r, located at $z = z_0$. Let numbers $E(z_0, r)$, $\beta(z_0)$ and $\gamma(z_0)$ be defined as follows.*

$$E(z_0, r) = \frac{|P^{(m)}(z_0)|}{m!} r^m - \sum_{j=0}^{m-1} \frac{|P^{(j)}(z_0)|}{j\,!} r^j - \sum_{j=m+1}^{n} \frac{|P^{(j)}(z_0)|}{j\,!} r^j, \tag{3.3}$$

$$\beta(z_0) = \max_{0 \le j < m} \left| \frac{m!\, P^{(j)}(z_0)}{j\,!\, P^{(m)}(z_0)} \right|^{1/(m-j)}, \tag{3.4}$$

$$\gamma(z_0) = \max_{m < j \le n} \left| \frac{m!\, P^{(j)}(z_0)}{j\,!\, P^{(m)}(z_0)} \right|^{1/(j-m)}. \tag{3.5}$$

If we can find a number r satisfying $0 < r < \frac{1}{2\,\gamma(z_0)}$ and $E(z_0, r) > 0$ then the disc $D(z_0, r)$ contains m close roots around z_0 and other $n-m$ roots are outside of $D(z_0, r)$.

Corollary 3 *If we can find a number r satisfying*

$$r < \frac{1}{2\gamma(z_0)}, \tag{3.6}$$

$$\frac{\beta(z_0)}{r} \le \frac{1 - 2\gamma(z_0)\, r}{2 - 3\gamma(z_0)\, r}, \tag{3.7}$$

then $D(z_0, r)$ contains m close roots of $P(z)$ and other $n-m$ roots are outside of $D(z_0, r)$.

Corollary 4 *If we can find a number r satisfying (3.6), (3.7) and*

$$\frac{\beta(z_0)}{r} \le \frac{1 - 4\gamma(z_0)\, r + 2\gamma(z_0)^2 r^2}{2 - 6\gamma(z_0)\, r + 3\gamma(z_0)^2 r^2}, \tag{3.8}$$

then $D(z_0, r)$ contains $m-1$ close roots of $P'(z)$ and other $n-m$ roots are outside of $D(z_0, r)$.

We consider Theorem 4. Expanding $P(z)$ at $z = z_0$ and moving the origin to $z = z_0$, we have $P(z_0{+}z) = P(z_0) + \frac{P^{(1)}(z_0)}{1!}\, z + \cdots + \frac{P^{(n)}(z_0)}{n!}\, z^n$. We next apply the scale transformation $z \mapsto z/\gamma(z_0)$ to $P(z_0{+}z)$, then we obtain

$$\begin{cases} \frac{\gamma(z_0)^m}{P^{(m)}(z_0)/m!}\, P(z_0 + z/\gamma(z_0)) \;=\; \tilde{c}_n z^n + \cdots + \tilde{c}_m z^m + \cdots + \tilde{c}_0, \\ \tilde{c}_m = 1, \qquad \max\{|\tilde{c}_{m+1}|,\, |\tilde{c}_{m+2}|,\, \cdots,\, |\tilde{c}_n|\} = 1, \\ \max\{|\tilde{c}_{m-1}|^{1/1}, |\tilde{c}_{m-2}|^{1/2}, \cdots, |\tilde{c}_0|^{1/m}\} = \gamma(z_0)\beta(z_0). \end{cases}$$

Thus, $\beta(z_0)\gamma(z_0)$ in Theorem 4 corresponds to e in Theorem 1.

Remembering the scale transformation $z \mapsto z/\gamma(z_0)$, we put $\gamma(z_0)\, r = R$ and $\beta(z_0)\gamma(z_0) = e$, then (3.7) becomes

$$\frac{e}{R} \le \frac{1-2R}{2-3R}.$$

Solving this inequality, we obtain $R_{\rm in} \le R \le R_{\rm out}$. (It is interesting to note that different theorems are used in [7] and [6] to derive the above same inequality: Yakoubsohn used Rouche's theorem and Terui-Sasaki utilized a famous theorem for bounding the roots.) Therefore, the smallest disc $D(z_0, R_{\rm in}/\gamma(z_0))$ satisfying (3.7) is equivalent to $D_{\rm in}$ in Theorem 2. On the other hand, condition (3.6) becomes $R < 1/2$, hence this condition is unnecessary because $R < R_{\rm out} < 1/2$. Similarly, condition (3.8) becomes

$$\frac{e}{R} \le \frac{1-4R+2R^2}{2-6R+3R^2}.$$

We can easily show that

$$\frac{1-2R}{2-3R} > \frac{1-4R+2R^2}{2-6R+3R^2} \quad \text{for} \quad 0 < R < 1/3.$$

This means that condition (3.7) is stronger than condition (3.8) for bounding R. Therefore, condition (3.8) is also unnecessary. Thus, Theorems 1 and 2 are more complete than Theorem 4 and its corollaries.

4 Variations of the theorem

4.1 For general polynomial

In **2**, we restricted $P(z)$ to be of a very special form; in particular, we assumed that the location of a cluster of close roots and m, the weight of the cluster, are known. We have no such information on polynomials given generally. The first step for separating clusters of close roots is to find the location, the spread and the weight of each cluster of close roots. We can get these informations by applying the approximate square-free decomposition to $P(z)$; see [4] or [3] for details. Furthermore, we can easily know the magnitude of the largest root $\zeta_{\max}$ of $P(z)$. Suppose we know that $P(z)$ has a cluster of close roots of weight m at $z = z_0$. Then, we perform the transformation

$$P(z) \mapsto \tilde{P}(z) = \beta P(z_0 + z/\gamma), \quad \gamma \simeq |\zeta_{\max}|, \tag{4.1}$$

where β and γ are determined to make $\tilde{P}(x)$ satisfy the conditions in (2.2). We may instead compute β and γ by formulas in (3.4) and (3.5), so long as m is known.

4.2 For polynomial having error terms

We can easily extend theorems 1 and 2 to polynomials with erroneous terms. Suppose $P(z)$ can be expressed as $P(z) = \hat{P}(z) + \Delta(z)$, where $\hat{P}(z)$ is a polynomial with exact coefficients,

$$\hat{P}(z) = \hat{c}_n z^n + \cdots + z^m + \hat{e}_{m-1} z^{m-1} + \cdots, \tag{4.2}$$

and coefficients $\hat{c}_i$ $(i > m)$ and $\hat{e}_j$ $(j < m)$ are bounded by small definite numbers $\delta_n, \ldots, \delta_0$ as

$$|c_i - \hat{c}_i| \le \delta_i \quad (i > m), \qquad |e_j - \hat{e}_j| \le \delta_j \quad (j < m). \tag{4.3}$$

Then, Theorems 1 and 2 are valid for $P(z)$ so long as the conditions in (2.2) are modified as

$$\begin{cases} \max\{\, |\hat{c}_n| - \delta_n, \,\cdots, \, |\hat{c}_{m+1}| - \delta_{m+1} \,\} = 1, \\ e \stackrel{\text{def}}{=} \max\{\, (|\hat{e}_{m-1}| + \delta_{m-1}), \, (|\hat{e}_{m-2}| + \delta_{m-2})^{1/2}, \,\cdots, \, (|e_0| + \delta_0)^{1/m} \,\} < 1/9. \end{cases} \tag{4.4}$$

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